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EXAMPLES FOR THE MOD p MOTIVIC COHOMOLOGY OF CLASSIFYING SPACES

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ABSTRACT. Let BG be the classifying space of a compact Lie group G. Some examples of computations of the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/p)$ are given, by comparing with $H^*(BG;\mathbb{Z}/p)$, $CH^*(BG)$ and $BP^*(BG)$.

1. INTRODUCTION

Let p be a prime number and k a subfield of the complex number field $\mathbb{C}$. Let k contain a primitive p-th root of unity. Given a scheme X of finite type over k, the mod p motivic cohomology $H^{*,*}(X;\mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X;\mathbb{Z}/p)$ has been defined by Suslin and Voevodsky ([Vo1], [Vo2]). When X is smooth, the subring $H^{2*,*}(X;\mathbb{Z}/p) = \bigoplus_n H^{2n,n}(X;\mathbb{Z}/p)$ is identified with the classical mod p Chow ring $CH^*(X)/p$ of algebraic cyles on X.

The inclusion $t_{\mathbb{C}} : k \subset \mathbb{C}$ induces a natural transformation (realization map) $t_{\mathbb{C}}^{m,n} : H^{m,n}(X;\mathbb{Z}/p) \to H^m(X(\mathbb{C});\mathbb{Z}/p)$, where $X(\mathbb{C})$ is the complex variety of $\mathbb{C}$-valued points of X. Let us write the coimage of $t_{\mathbb{C}}^{*,*}$ as

$$h^{*,*}(X;\mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}}^{m,n}). \tag{1.1}$$

It is known that there is an element $\tau \in H^{0,1}(Spec(k);\mathbb{Z}/p)$ with $t_{\mathbb{C}}^{*,*}(\tau) = 1$. Then we have the bigraded $\mathbb{Z}/p[\tau]$-algebra monomorphism

$$h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau,\tau^{-1}] \tag{1.2}$$

where the bidegree of $x \in H^n(X(\mathbb{C});\mathbb{Z}/p)$ is given by (n,n). If $k = \mathbb{C}$ and the Beilinson-Lichtenbaum condition [Vo2] is satisfied for p, then we also have the injection $H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X;\mathbb{Z}/p)$.

When $x \in H^{m,n}(X;\mathbb{Z}/p)$, define the weight of x by $w(x) = 2n - m$. Clearly $w(x) = 0$ if and only if $x \in CH^*(X)/p$. Voevodsky has extended the Steenrod algebra A_p^* of cohomology operations to the case of motivic cohomology. Among them, we have the Milnor primitive operation

$$Q_i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p),$$

so that it is sent to the usual Milnor operation Q_i by the realization map $t_{\mathbb{C}}^*$. Hence $w(Q_i) = -1$, and the Q_i $(0 \le i)$ form an exterior algebra $\Lambda(Q_0, Q_1, ...) \subset A_p^*$ also for the motivic cohomology. To simplify the notation, let us write the exterior algebra $Q(n) = \Lambda(Q_0, ..., Q_n)$ for $n \ge 0$ and $Q(-1) = \mathbb{Z}/p$.

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In this paper we are mainly concerned with the following case. For $n \geq -1$, let G_n be a $\mathbb{Z}/p$-module and $Q(n)G_n$ the free $Q(n)$-module generated by G_n. Moreover, the scheme X satisfies the assumption that there is a $\mathbb{Z}/p$-module injection

$$(1.3)\quad j_{\mathbb{C}} : H^*(X(\mathbb{C});\mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \quad \text{with } j_{\mathbb{C}}^{-1}(Q_0...Q_nG_n) \subset Im(t_{\mathbb{C}}^{2*,*})$$

such that $p_n j_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \to Q(n)G_n$ is the $Q(n)$-module map and $p'_n p_n j_{\mathbb{C}} : H^*(X(\mathbb{C}) : \mathbb{Z}/p) \to Q_0...Q_{n-1}G_n$ is a surjection for each n, where $p_n : \bigoplus Q(n)G_n \to Q(n)G_n$ and $p'_n : Q(n)G_n \to Q_0...Q_{n-1}G_n$ are the projections. (We do not assume a $Q(n)$-module structure on the right-hand side module in (1.3).)

We take the weight on the right-hand side by putting $w(x) = n+1$ for every $x \in G_n$ (simply write $w(G_n) = n+1$), so that $w(Q_0...Q_nx) = 0$. Then we get the injection of bigraded $\mathbb{Z}/p$-modules

$$(1.4)\qquad j : h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow \bigoplus_{n=-1}^{\infty} Q(n)G_n \otimes \mathbb{Z}/p[\tau]$$

such that the composition $(p_n \otimes \mathbb{Z}/p[\tau])j : h^{*,*}(X;\mathbb{Z}/p) \to Q(n)G_n \otimes \mathbb{Z}/p[\tau]$ is the bigraded $Q(n)$-module map.

The above argument has its counterpart in the BP-theory of $X(\mathbb{C})$. As we know, $BP^*(-)$ is the cohomology theory with the coefficient ring $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$, $|v_i| = -2(p^i - 1)$. Let us write $BP^*/(p, v_1, ..., v_{m-1})$ as $P(m)^*$. The Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(X(\mathbb{C})) \otimes BP^* \Longrightarrow BP^*(X(\mathbb{C}))$$

has the differential

$$(1.5)\qquad d_{2p^i-1}(x) = Q_i(x) \otimes v_i \qquad \mathrm{mod}(M_i),$$

where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, ..., v_{i-1})E_2^{*,*}$ We assume here that nonzero differentials are all of the form (1.5) and that $H^*(X(\mathbb{C}))$ has no higher p-torsion. Then we easily see that (1.3) implies

$$(1.6)\qquad E_\infty^{*,*} \cong \bigoplus_{n=-1}^{\infty} P(n+1)^*\tilde{G}_n \ \oplus\ B \qquad \text{with} \quad \tilde{G} = Q_0...Q_nG_n,$$

where $P(n+1)^*\tilde{G}_n$ is the free $P(n+1)^*$-module generated by elements in $\tilde{G}_n$ and B is the BP^*-submodule of $E_\infty^{*,*}$ of generators in Ideal$(p, v_1, ...)E_2^{*,*}$. Conversely, by the same assumption, if $\tilde{G}_n \subset \mathrm{Im}(t_{\mathbb{C}}^{2*,*})$, then the isomorphism (1.6) implies the existence of the injections $j_{\mathbb{C}}$ in (1.3) and so j in (1.4).

Let $\rho : BP(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p \to H^*(X(\mathbb{C});\mathbb{Z}/p)$ be the Thom map. Then (1.6) and $\tilde{G}_n \subset \mathrm{Im}(t_{\mathbb{C}}^{2*,*})$ imply that

$$\mathrm{Im}(t_{\mathbb{C}}^{2*,*}) = \mathrm{Im}(\rho) \cong \bigoplus_{n=-1}^{\infty} \tilde{G}_n \subset BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p.$$

More generally, B. Totaro [To1], [To2] constructed the modified cycle map

$$(1.7)\qquad \tilde{cl}^* : CH^*(X)/p \to BP^*(X(\mathbb{C})) \otimes_{BP^*} \mathbb{Z}/p$$

in such a way that the composition $\rho\tilde{cl}^*$ is the realization map $t_{\mathbb{C}}^{2*,*}$. If a BP^*-module generator of B in (1.6) is represented by transfer of a Chern class, then this element gives a nonzero element in $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ by the modified cycle map $\tilde{cl}^*$. Using this argument, Totaro found nonzero elements in $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ when X is the classifying space $BSO(4)$.

The motivic cohomology of the classifying space is defined as follows. Let G be a linear algebraic group over k. Let V be a representation of G such that G acts freely on $V - S$ for some closed subset S. Then $(V - S)/G$ exists as a quasi-projective variety over k. Following Totaro [To1] and Voevodsky, define

$$H^{*,*}(BG;\mathbb{Z}/p) = \lim_{dim(V),codim(S)\to\infty} H^{*,*}((V-S)/G;\mathbb{Z}/p). \tag{1.8}$$

The topological space $BG(\mathbb{C}) = \lim((V-S)/G)(\mathbb{C})$ is the usual classifying space BG. Hence we write the $\mathbb{C}$-value points $BG(\mathbb{C})$ simply as BG.

We will show that the isomorphism (1.6) is satisfied when $X = BG$ for the following cases: $O(n), SO(4), D_8, G_2, Spin(7)$ for $p = 2$, PGL_3, F_4 for $p = 3$ and the extraspecial p-group p_+^{1+2} of order p^3 and of exponent p for odd primes. (However note that $H^*(Bp_+^{1+2})$ has p^2-torsion.)

Hence we will prove (1.4) for these BG. Moreover, when $k = \mathbb{C}$ and $G = O(3)$ for $p = 2$, PGL_3 for $p = 3$, p_+^{1+2} and $(\mathbb{Z}/p)^n$ for all primes, we will show that

$$h^{*,*}(BG;\mathbb{Z}/p) \cong \bigoplus Q(n)G_n \otimes \mathbb{Z}/p[\tau]. \tag{1.9}$$

S. Wilson [RWY] first constructed the decomposition (1.3) so that $j_{\mathbb{C}}$ is an isomorphism for $X = BO(n)$, and next computed $BP^*(BO(n))$. However, it is unknown whether j in (1.4) is an isomorphism or not for $X = BO(n)$, $n \geq 4$.

The contents of this paper are as follows. The aim of §§2 and 3 is a short introduction to motivic cohomology for algebraic topologists unfamiliar with it. In these sections, we concentrate on the computation of $H^*(B(\mathbb{Z}/p)^n;\mathbb{Z}/p)$. In §4, we deal with the study of $h^{*,*}(X;\mathbb{Z}/p)$, making no use of $BP^*(BG)$ but Milnor's operation Q_i instead. In §5, we give an account of $h^{*,*}(BG;\mathbb{Z}/p)$ expressed in term of $BP^*(BG)$. Also in this section we give some results on $\mathrm{Ker}(t_{\mathbb{C}}^{*,*})$. The motivic cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}/p(n), n)$ is studied in §6. In §7, we give some comments on algebraic cobordism theory and algebraic BP-theory.

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2. Chow ring, Milnor K-theory, étale cohomology

We use the category Spc of (algebraic) spaces, along with schemes A, their quotients A_1/A_2 and $\mathrm{colim}(A_\alpha)$, all defined by Voevodsky [Vo2], [MoVo]. Here schemes are defined over a field k with $ch(k) = 0$. The motivic cohomology is the double indexed cohomology defined by Suslin and Voevodsky, directly related with the Chow ring and Milnor K-theory.

(CH) For a smooth scheme X we have $H^{2n,n}(X) \cong CH^n(X)$, the classical Chow group of codim n cycles on X.

(MK) $H^{n,n}(Spec(k)) \cong K_n^M(k)$, the Milnor K-group for the field k.

For a smooth variety X with $dim(X) = n$, the Chow ring is the sum $CH^*(X) = \bigoplus_i CH^i(X)$, where

$$CH^i(X) = \{(n-i) \text{ cycles in } X\}/(\text{ rational equivalence}).$$

Here the rational equivalence $a \equiv b$ is defined if there is a codimension i subvariety W in $X \times \mathbb{P}^1$ such that $a = p_*f^*(0)$ and $b = p_*f^*(1)$, where $\mathbb{P}^1$ is the projective line and p (resp. f) is the projection on the first (resp. second) factor.

For $k = \mathbb{C}$, if X has a cellular decomposition, i.e., $X = X_n \supset X_{n-1} \supset ... \supset X_0$ with $X_i - X_{i-1} = \bigcup \mathbb{A}^{n_{ij}}$, where $\mathbb{A}^{n_{ij}}$ is the affine space of dimension n_{ij}, then $CH^*(X) \cong H^*(X(\mathbb{C}))$, the singular cohomology theory of $\mathbb{C}$-rational points of X. For example, $CH^*(\mathbb{P}^n) \cong H^*(\mathbb{CP}^n)$ for projective spaces $\mathbb{P}^n$. Since Spc contains *colimit*, we can consider the infinite projective space $\mathbb{P}^\infty = B\mathbb{G}_m$ and the infinite lens space $\lim_n(\mathbb{A}^n - \{0\}/\mathbb{Z}/p) = L_p^\infty = B\mathbb{Z}/p$. The Chow rings of classifying spaces of abelian groups are given in [To1]:

(2.1)
$$CH^*(\mathbb{P}^\infty) \cong H^{2*,*}(\mathbb{P}^\infty) \cong \mathbb{Z}[y], \quad CH^*(B\mathbb{Z}/p) \cong H^{2*,*}(B\mathbb{Z}/p) \cong \mathbb{Z}[y]/(py),$$

with $deg(y) = (2,1)$. For products of these spaces we have

(2.2)
$$CH^*((\mathbb{P}^\infty)^n) \cong \mathbb{Z}[y_1, ..., y_n],$$

(2.3)
$$CH^*((B\mathbb{Z}/p)^n) \cong \mathbb{Z}[y_1, ..., y_n]/(py_1, ..., py_n).$$

Here note that $CH^*(X) \ncong H^{even}(X(\mathbb{C}))$ for the last case. Even if $H^*(X(\mathbb{C}))$ is generated by even dimensional elements, there are cases that $CH^*(X) \ncong H^*(X(\mathbb{C}))$, e.g., K3-surfaces have the cohomology $H^2(X(\mathbb{C})) \cong \mathbb{Z}^{22}$, but there is a K3-surface such that $CH^1(X) \cong \mathbb{Z}^i$ for each $1 \le i \le 20$.

Milnor K-theory is the graded ring $\bigoplus_n K_n^M(k)$ defined by $K_n^M(k) = (k^*)^{\otimes n}/J$, where the ideal J is generated by elements $a \otimes (1-a)$ for $a \in k^* - \{1\}$. Here the addition of k^* is given by the multiplication in the field k. Hence $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^*$. Hilbert's Theorem 90, which essentially says that the Galois cohomology $H^1(G(k_s/k); k_s^*) = 0$, implies the isomorphism $K_1^M(k)/p \cong k^*/(k^*)^p \cong H^1(G(k_s/k); \mathbb{Z}/p)$ for $1/p \in k$. Similarly we can define a map (the norm residue map) for any extension F of k of finite type,

(BK)
$$K_n^M(F)/p \to H^n(G(F_s/F); \mu_p^{\otimes n}),$$

where $\mu_p^{\otimes n}$ is the discrete $G(F_s/F)$-module of n-th tensor power of the group of p-roots of 1. The Bloch-Kato conjecture is that this map is an isomorphism for all field k, and the Milnor conjecture is its $p = 2$ case. This conjecture is solved when $n = 2$ by Merkurjev and Suslin [MeSu], and for $p = 2$ by Voevodsky [Vo1].

Notice that $H^n(G(k_s/k); \mu_p^{\otimes n}) \cong H_{et}^n(Spec(k), \mu_p^{\otimes n})$, the étale cohomology of the point. The étale cohomology $H_{et}^*(X; \mathbb{Z}/p)$ has the following properties:

(E.1) If k contains a primitive p-th root of 1, then there is the additive isomorphism
$$H_{et}^m(X, \mu_p^{\otimes n}) \cong H_{et}^m(X; \mathbb{Z}/p).$$

(E.2) For smooth X over $k = \mathbb{C}$,
$$H_{et}^m(X; \mathbb{Z}/p^N) \cong H^m(X(\mathbb{C}); \mathbb{Z}/p^N) \quad \text{for all } N \ge 1.$$

The last cohomology is the usual mod p ordinary cohomology of $\mathbb{C}$-rational points of X. Of course $H_{et}^*(Spec(\mathbb{C}); \mathbb{Z}/p) \cong \mathbb{Z}/p$. It is known that
$$K_*^M(\mathbb{R})/2 \cong H_{et}^*(Spec(\mathbb{R}); \mathbb{Z}/2) \cong \mathbb{Z}/2[\rho]$$
with $deg(\rho) = 1$ for the real number field $\mathbb{R}$. Let F_v be a local field with residue field k_v of $ch(k_v) \ne 2$. Then $K_*^M(F_v)/2 \cong H_{et}^*(Spec(F_v); \mathbb{Z}/2) \cong \Lambda(\alpha, \beta)$ with $deg(\alpha) = deg(\beta) = 1$. Thus we know that $\bigoplus_m H^{m,m}(pt; \mathbb{Z}/2)$ for these cases.

3. The realization map

In this section we consider the relation to the usual ordinary cohomology. Let R be $\mathbb{Z}$ or $\mathbb{Z}/p$. The motivic cohomology has the following properties [Vo2].

(C1) $H^{*,*}(X;R)$ is a bigraded ring natural in X.

(C2) When $k \subset \mathbb{C}$, there are maps (realization maps)

$$t_{\mathbb{C}}^{m,n} : H^{m,n}(X;R) \to H^m(X(\mathbb{C});R)$$

which sum up to $t_{\mathbb{C}}^{*,*} = \bigoplus_{m,n} t_{\mathbb{C}}^{m,n}$, the natural ring homomorphism.

(C3) There are the (Bockstein, reduced powers) operations

$$\beta : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+1,*}(X;\mathbb{Z}/p),$$
$$P^i : H^{*,*}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i,*+(p-1)i}(X;\mathbb{Z}/p),$$

which commutes with the realization map $t_{\mathbb{C}}$ when $k \subset \mathbb{C}$.

(C4) For the projective space $\mathbb{P}^n$, there is an isomorphism

$$H^{*,*}(X \times \mathbb{P}^n/\mathbb{P}^{n-1};R) \cong H^{*,*}(X;R)\{1,y'\}$$

with $deg(y') = (2n,n)$ and $t_{\mathbb{C}}(y') \neq 0$ for $k \subset \mathbb{C}$.

We recall the Lichtenbaum motivic cohomology [Vo2]. Lichtenbaum defined the similar cohomology $H_L^{*,*}(X;R)$ by using the étale topology, while $H^{*,*}(X;R)$ is defined using the Nisnevich topology. Since Nisnevich covers are restricted étale covers, there is the natural map $H^{*,*}(X;R) \to H_L^{*,*}(X;R)$. We say that the $B(n,p)$ condition holds if

$$H^{m,n}(X;Z_{(p)}) \cong H_L^{m,n}(X;Z_{(p)}) \quad \text{for all } m \leq n+1$$

and all smooth X. The Beilinson-Lichtenbaum conjecture is that $B(n,p)$ holds for all n and p. It is known that the condition $B(n,p)$ is equivalent to the Bloch-Kato conjecture (BK) for degree n and prime p. Hence $B(n,p)$ holds for $n \leq 2$ or $p = 2$. Moreover, Suslin and Voevodsky have proved

(L-E) $$H_L^{m,n}(X;\mathbb{Z}/p) \cong H_{et}^m(X;\mu_p^{\otimes n}).$$

Now we compute $H^{*,*}(pt;\mathbb{Z}/p) = H^{*,*}(Spec(k);\mathbb{Z}/p)$. For a smooth X, the following dimensional condition is known:

(C5) For a smooth X, if $H^{m,n}(X;R) \ncong 0$, then

$$m \leq n + dim(X), \qquad m \leq 2n \text{ and } m \geq 0.$$

For the rest of this paper, we assume that k contains a primitive p-th root of 1 and $B(n,p)$ holds for all n, but $X = Spec(k)$. Then

$$H^{m,n}(pt;\mathbb{Z}/p) \cong H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mathbb{Z}/p) \qquad \text{if } m \leq n,$$

and $H^{m,n}(pt;\mathbb{Z}/p) \cong 0$ for $m > n$. Let $\tau \in H^{0,1}(pt;\mathbb{Z}/p)$ be the element corresponding to a generator of $H_{et}^0(Spec(k);\mu_p) \cong H_{et}^0(Spec(k);\mathbb{Z}/p)$. Then we get the isomorphism

$$H^{*,*}(Spec(k);\mathbb{Z}/p) \cong H_{et}^*(Spec(k);\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau]$$

since $\tau : H_{et}^m(pt;\mu_p^{\otimes n}) \cong H_{et}^m(pt;\mu_p^{\otimes(n+1)})$. In particular, for the real number field $\mathbb{R}$ and a local field F_v with the residue field k_v of $ch(k_v) \neq 2$ we have

(3.1) $$H^{*,*}(Spec(\mathbb{R});\mathbb{Z}/2) \cong \mathbb{Z}/2[\rho,\tau] \quad \text{with } deg(\rho) = (1,1),$$

(3.2) $$H^{*,*}(Spec(F_v);\mathbb{Z}/2) \cong \mathbb{Z}/2[\tau] \otimes \Lambda(\alpha,\beta) \quad \text{with } deg(\alpha) = deg(\beta) = (1,1).$$

For $k=\mathbb{C}$, we know that $K_n^M(\mathbb{C})/p \cong 0$ for $n>0$, and hence

$$H^{*,*}(Spec(\mathbb{C});\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \quad \text{with } deg(\tau)=(0,1). \tag{3.3}$$

When $k=\mathbb{C}$, if the $B(n,p)$ condition holds for X, then it is immediate that

$$[\tau^{-1}]H^{*,*}(X;\mathbb{Z}/p) \cong H^*(X(\mathbb{C});\mathbb{Z}/p)\otimes \mathbb{Z}/p[\tau,\tau^{-1}], \tag{3.4}$$

where the degree is defined by $deg(x)=(m,m)$ if $x\in H^m(X(\mathbb{C});\mathbb{Z}/p)$.

Next we compute the cohomology of $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$. For any (algebraic) map $f:X\to Y$ in the category Spc, we can construct the cofiber sequence

$$X\to Y\to cone(f)=Y/X,$$

which induces the long exact sequence (Voevodsky [Ve2])

$$H^{*,*}(X;R)\leftarrow H^{*,*}(Y;R)\leftarrow H^{*,*}(Y/X:R)\leftarrow H^{*-1,*}(X;R). \tag{3.5}$$

In particular, we get the Mayer-Vietoris, Gysin and blow-up long exact sequences.

By the cofiber sequence $\mathbb{P}^{n-1}\to\mathbb{P}^n\to\mathbb{P}^n/\mathbb{P}^{n-1}$ and (C4), we can inductively see that

$$H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\cong H^{*,*}(pt;\mathbb{Z}/p)\otimes\mathbb{Z}/p[y]/(y^{n+1}) \quad \text{with } deg(y)=(2,1). \tag{3.6}$$

When $k=\mathbb{C}$, since $B(1,p)$ always holds, $H^{1,1}(L_p^n;\mathbb{Z}/p)\cong H^1(L_p^n;\mathbb{Z}/p)$. Hence there is an element $x'\in H^{1,1}(L_p^n;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x')=x\in H^1(L_p^n;\mathbb{Z}/p)$. This also holds for general k [Vo3]. The lens space is identified with the sphere bundle associated with the line bundle

$$(\mathbb{A}^n-\{0\})\times_{(\mathbb{A}-\{0\})}\mathbb{A}\to(\mathbb{A}^n-\{0\})/(\mathbb{A}-\{0\})=\mathbb{P}^n.$$

Here $(\mathbb{A}^n-\{0\})\times_{(\mathbb{A}-\{0\})}\mathbb{A}$ is the identification such that $(z_i,z)\sim(a^{-1}z_i,a^pz)\in(\mathbb{A}^n-\{0\})\times\mathbb{A}$ for $(z_i)\in\mathbb{A}^n,\ z\in\mathbb{A},\ a\in\mathbb{A}-\{0\}$. Hence we get the cofibering $L_p^n\to\mathbb{P}^n\xrightarrow{\times p}\mathbb{P}^n$. Thus we get the additive isomorphism $H^{*,*}(L_p^n;\mathbb{Z}/p)\cong H^{*,*}(\mathbb{P}^n;\mathbb{Z}/p)\{1,x\}$. This induces the ring isomorphism for $p=odd$

$$H^{*,*}(L_p^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y]/(y^{n+1})\otimes\Lambda(x)\otimes H^{*,*}(pt;\mathbb{Z}/p) \quad \text{with } deg(x)=(1,1). \tag{3.7}$$

However, note that when $p=2$ we get $x^2=y\tau+x\rho$ [Vo3], where $\rho\in H^{1,1}(pt;\mathbb{Z}/p)\cong k^*/k^{2*}$ represents -1. (Hence $\rho=0$ when $\sqrt{-1}\in k^*$.) This is proved by the well-known fact that $\{a,a\}=\{a,-1\}$ in the Milnor K-theory $K_2^M(k)$.

We say that a space X satisfies the Künneth formula for a space Y if

$$H^{*,*}(X\times Y;\mathbb{Z}/p)\cong H^{*,*}(X;\mathbb{Z}/p)\otimes_{H^{*,*}(pt;\mathbb{Z}/p)}H^{*,*}(Y;\mathbb{Z}/p).$$

By the above cofiber sequences, we can easily see that $\mathbb{P}^\infty$ and $B\mathbb{Z}/p$ satisfy the Künneth formula for all spaces. In particular, we have the ring isomorphisms

$$H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes H^{*,*}(pt;\mathbb{Z}/p), \tag{3.8}$$

$$H^{*,*}((B\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong\mathbb{Z}/p[y_1,...,y_n]\otimes\Lambda(x_1,...,x_n)\otimes H^{*,*}(pt;\mathbb{Z}/p) \tag{3.9}$$

(when $p=2$, $x_i^2=y_i\tau+x_i\rho$).

This fact is used to define the reduced power operation P^i in (C3). Since a Sylow p-subgroup of the symmetric group S_p of p letters is isomorphic to $\mathbb{Z}/p$, we have the isomorphism

$$H^{*,*}(BS_p;\mathbb{Z}/p)\cong H^{*,*}(B\mathbb{Z}/p;\mathbb{Z}/p)^{F_p^*}\cong\mathbb{Z}/p[Y]\otimes\Lambda(W)\otimes H^{*,*}(pt;\mathbb{Z}/p),$$

identifying $Y = y^{p-1}$ and $W = xy^{p-2}$. If X is smooth (and suppose p is odd, to simplify arguments), we can define the reduced powers (of Chow rings) as follows. Consider maps

$$\begin{aligned} &H^{2*,*}(X;\mathbb{Z}/p) \xrightarrow{i_!} H^{2p*,p*}(X^p \times_{S_p} ES_p) \\ &\xrightarrow{\Delta^*} H^{*,*}(X \times BS_p;\mathbb{Z}/p) \cong H^{*,*}(X;\mathbb{Z}/p) \otimes_{H^{*,*}(pt;\mathbb{Z}/p)} H^{*,*}(BS_p;\mathbb{Z}/p), \end{aligned}$$

where $i_!$ is the Gysin map for the p-th external power, and Δ is the diagonal map. For $deg(x) = (2n, n)$, the reduced powers are defined as

$$\Delta^* i_!(x) = \sum P^i(x) \otimes Y^{n-i} + \beta P^i(x) \otimes WY^{n-i-1}. \tag{3.10}$$

Hence $deg(P^i) = deg(Y^i) = deg(y^{i(p-1)}) = (2i(p-1), i(p-1))$. Voevodsky defined $i_!$ for nonsmooth X also, and by using suspensions maps he defined reduced powers for all degree elements in $H^{*,*}(X;\mathbb{Z}/p)$ for all X [Vo3].

Moreover, we can see (Hu-Kříž [HK]) that

$$H^{*,*}(BGL_n;\mathbb{Z}/p) \cong \mathbb{Z}/p[c_1, ..., c_n] \otimes H^{*,*}(pt;\mathbb{Z}/p), \tag{3.11}$$

where the Chern class c_i with $deg(c_i) = (2i, i)$ is identified with the elementary symmetric polynomial in $H^{*,*}((\mathbb{P}^\infty)^n;\mathbb{Z}/p)$. So we can define the Chern class $\rho^*(c_i) \in H^{2*,*}(BG;\mathbb{Z}/p)$ for each representation $\rho : G \to GL_n$.

4. $H^{*,*}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}})$ and the operation Q_i

In this section we assume that X is smooth and $k = \mathbb{C}$. Even in this case the motivic cohomology $H^{*,*}(X;\mathbb{Z}/p)$ seems difficult, in general. Hence we consider a bigraded ring which is computable only by using the algebraic topology of $H^*(X(\mathbb{C});\mathbb{Z}/p)$. Define a bidegree algebra by

$$h^{*,*}(X;\mathbb{Z}/p) = \bigoplus_{m,n} H^{m,n}(X;\mathbb{Z}/p)/\operatorname{Ker}(t_{\mathbb{C}}^{m,n}). \tag{4.1}$$

Since $t_{\mathbb{C}}^{*,*}(\tau) = 1$, it is almost immediate that there is the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras

$$h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau, \tau^{-1}],$$

where the bidegree of $x \in H^n(X(\mathbb{C});\mathbb{Z}/p)$ is (n, n). (This also holds when $k \subset \mathbb{C}$ and k has a primitive p-th root of 1.)

Suppose the $B(n, p)$ condition holds. By the isomorphisms (B, p), (L-E), (E1) and (E2), we have

$$H^{n,n}(X;\mathbb{Z}/p) \cong H_L^{n,n}(X;\mathbb{Z}/p) \cong H_{et}^n(X;\mu_p^{\otimes n}) \cong H_{et}^n(X;\mathbb{Z}/p) \cong H^n(X(\mathbb{C});\mathbb{Z}/p).$$

Hence we get the injection of bidegree $\mathbb{Z}/p[\tau]$-algebras

$$H^*(X(\mathbb{C});\mathbb{Z}/p) \otimes \mathbb{Z}/p[\tau] \hookrightarrow h^{*,*}(X;\mathbb{Z}/p).$$

Thus there exist a $\mathbb{Z}/p$-basis $\{a_I\}$ of $H^*(X(\mathbb{C});\mathbb{Z}/p)$ and a $|\frac{1}{2}a_I| \geq t_I \geq 0$ such that

$$h^{*,*}(X;\mathbb{Z}/p) \cong \bigoplus_I \mathbb{Z}/p[\tau]\{\tau^{-t_I} a_I\}.$$

Remark. Let $F_i = \mathrm{Im}(\bigoplus_m t_{\mathbb{C}}^{m,i})$. When the $B(n,p)$ condition is satisfied, we have $\bigcup_i F_i = H^*(X(\mathbb{C});\mathbb{Z}/p)$. We also have the interesting bigraded ring

$$grH^*(X(\mathbb{C});\mathbb{Z}/p) = \bigoplus F_{i+1}/F_i \cong h^{*,*}(X;\mathbb{Z}/p)/(\mathrm{Im}\,\tau),$$

so that $\mathbb{Z}/p[\tau]\otimes grH^*(X(\mathbb{C});\mathbb{Z}/p)$ is additively isomorphic to $h^{*,*}(X;\mathbb{Z}/p)$, while the ring structures are different.

Here we recall the Milnor primitive operations $Q_0=\beta$ and $Q_i=[Q_{i-1},P^{p^{i-1}}]$:

$$Q_i : H^{*,*}(X;\mathbb{Z}/p)\to H^{*+2p^i-1,*+p^i-1}(X;\mathbb{Z}/p),$$

which is derivative, $Q_i(xy)=Q_i(x)y+xQ_i(y)$. Note also that $Q_i(\tau)=0$, because of the dimension of $H^{*,*}(pt;\mathbb{Z}/p)\cong\mathbb{Z}/p[\tau]$.

Lemma 4.1. *If $0\neq Q_{i_1}...Q_{i_s}x\in H^{2*,*}(X;\mathbb{Z}/p)$, then x is a $\mathbb{Z}/p[\tau]$-module generator.*

Proof. If $x=x'\tau$, then $\tau Q_{i_1}...Q_{i_s}(x')\neq 0$. But

$$Q_{i_1}...Q_{i_s}(x')=0\in H^{2*,*-1}(X;\mathbb{Z}/p),$$

since $H^{m,n}(X;\mathbb{Z}/p)=0$ for $m>2n$. □

Define the weight by $w(x)=2n-m$ for an element $x\in H^{m,n}(X;\mathbb{Z}/p)$, so that $w(x')=0$ for $x'\in CH^*(X)/p$. Of course we get $w(xy)=w(x)+w(y)$, $w(P^ix)=w(x)$ and $w(Q_i(x))=w(x)-1$.

Corollary 4.2. *Suppose that $B(n,p)$ holds. If $x\in H^n(X(\mathbb{C});\mathbb{Z}/p)$ and $Q_{i_1}...Q_{i_n}(x)\neq 0$, then there is a $\mathbb{Z}/p[\tau]$-module generator $x'\in H^{n,n}(X;\mathbb{Z}/p)$ so that $t_{\mathbb{C}}(x')=x$ and, for each $0\le k\le n$, $Q_{i_1}...Q_{i_k}(x')$ is also a $\mathbb{Z}/p[\tau]$-module generator of $H^{*,*}(X;\mathbb{Z}/p)$.*

Proof. By the $B(n,p)$ condition, $t_{\mathbb{C}}^{n,n}:H^{n,n}(X;\mathbb{Z}/p)\cong H^n(X(\mathbb{C});\mathbb{Z}/p)$. Hence there is an element $x'\in H^{n,n}(X;\mathbb{Z}/p)$ with $t_{\mathbb{C}}(x')=x$. This means $w(x')=n$ and $w(Q_{i_1}...Q_{i_n}(x))=0$. From the above lemma, we get the corollary. □

Lemma 4.3. *Suppose that $B(n,p)$ holds. If there is an $s>0$ with $p^sH^{n+1}(X(\mathbb{C}))_{(p)}\subset t_{\mathbb{C}}(H^{n+1,n}(X)_{(p)})$, then*

$$\mathrm{Im}(H^{n+1}(X(\mathbb{C}))\to H^{n+1}(X(\mathbb{C});\mathbb{Z}/p))=\mathrm{Im}((H^{n+1,n}(X)\to H^{n+1}(X(\mathbb{C});\mathbb{Z}/p)).$$

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
H_L^{n+1,n}(X) & \xrightarrow{(1)} & H_L^{n+1,n}(X;\mathbb{Z}/p^N) & \longrightarrow & H_L^{n+2,n}(X) & \xrightarrow{p^N} & H_L^{n+2,n}(X)\\
{\scriptstyle (2)}\downarrow & & \cong\downarrow & & \downarrow & & \downarrow\\
H^{n+1}(X(\mathbb{C})) & \xrightarrow{(3)} & H^{n+1}(X(\mathbb{C});\mathbb{Z}/p^N) & \longrightarrow & H^{n+2}(X(\mathbb{C})) & \xrightarrow{p^N} & H^{n+2}(X(\mathbb{C}))
\end{array}$$

where $H^*(-)$ means $H^*(-;\mathbb{Z})_{(p)}$ and the rows are exact.

Let $H^{n+i}(X(\mathbb{C}))\cong F_i\oplus T_i$ and $H_L^{n+i,n}(X)\cong F_i'\oplus T_i'\oplus D_i$, where F_i,F_i' are free, T_i,T_i' are non-p-divisible torsion and D_i are p-divisible submodules. Take N and s so that $p^N>p^s>|T_i|,|T_i'|$ for $i=1,2$. Hence $H_L^{n+1,n}(X;\mathbb{Z}/p^N)\cong H^{n+1}(X(\mathbb{C});\mathbb{Z}/p^N)\cong F_1/p^N\oplus T_1\oplus T_2$.

By the $B(n,p)$ condition, $H^{n+1,n}(X)\cong H_L^{n+1,n}(X)$, and the map (2) is identified with the realization map. So $p^s(F_1\oplus T_1)=p^sF_1\subset \mathrm{Image}(2)$. Therefore there is the quotient map $F_1/p^s\oplus T_1\oplus T_2\to \mathrm{Coker}(1)$. On the other hand,

$\mathrm{Ker}(p^N)|H_L^{n+2,n}(X) \cong (\mathrm{Ker}(p^N)|D_2) \oplus T_2' \cong (\mathbb{Z}/p^N)^k \oplus T_2'$. Hence if $k \neq 0$, then it is a contradiction to $\mathrm{Ker}(p^N) = \mathrm{Coker}(1)$. Hence we get $\mathrm{Coker}(1) \cong T_2'$ and hence $\mathrm{Im}(3)(2) = F_1/p^N \oplus T_1$. □

Corollary 4.4. *Suppose that $B(n,p)$ holds and $t_{\mathbb{C}}^{n+1,n} \otimes \mathbb{Q} : H^{n+1,n}(X) \otimes \mathbb{Q} \to H^{n+1}(X(\mathbb{C})) \otimes \mathbb{Q}$ is epic. If $x \in \mathrm{Im}(H^{n+1}(X(\mathbb{C})) \to H^{n+1}(X(\mathbb{C}); \mathbb{Z}/p))$ and $Q_{i_1}...Q_{i_{n-1}}(x) \neq 0$, then there is an element $x' \in H^{n+1,n}(X)_{(p)}$ so that $t_{\mathbb{C}}(x') = x$ and, for each $0 \leq k \leq n-1$, $Q_{i_1}...Q_{i_k}(x)$ is also a $\mathbb{Z}/p[\tau]$-module generator of $H^{*,*}(X; \mathbb{Z}/p)$.*

Here we mention the case $n = 1$. Totaro showed [To2] that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$ for any complex algebraic group G. Hence $CH^1(BG) \to H^2(BG)$ is epic; indeed, he also showed that this map is an isomorphism. As for $K3$-surfaces, $CH^*(X) \otimes \mathbb{Q} \to H^*(X(\mathbb{C})) \otimes \mathbb{Q}$ is not epic and $H_L^{3,1}(X)$ contains p-divisible elements.

Now we consider some examples. The mod 2 cohomology of $BO(n)$ is $H^*(BO(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, ..., w_n]$, where the Stiefel-Whitney class w_i restricts the elementary symmetric polynomial in $H^*(B(\mathbb{Z}/2)^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, ..., x_n]$. Each element w_i^2 is represented by the Chern class c_i of the induced representation $O(n) \subset U(n)$. Hence $c_i \in CH^*(BO(n); \mathbb{Z}/2) = H^{2*,*}(BO(n); \mathbb{Z}/2)$.

Proposition 4.5. *$h^{*,*}(BO(n); \mathbb{Z}/2) \supset \mathbb{Z}/2[c_1, ..., c_n] \otimes \Delta(w_1, ..., w_n) \otimes \mathbb{Z}/2[\tau]$, where $deg(c_i) = (2i, i)$, $deg(w_i) = (i, i)$ and $w_i^2 = \tau^i c_i$.*

Since $Q_{i-1}...Q_0(w_i) \neq 0$, each w_i is a $\mathbb{Z}/2[\tau]$-module generator. However, even $h^{*,*}(BO(n); \mathbb{Z}/2)$ seems very complicated. Consider the case $X = BO(3)$. The cohomology operations act by

$$w_2 \xrightarrow{Sq^1} w_1w_2 + w_3 \xrightarrow{Sq^2} w_2w_1^3 + w_1^2w_3 + w_1w_2^2 + w_2w_3 \xrightarrow{Sq^1} w_1^2w_2^2 + w_3^2,$$
$$w_3 \xrightarrow{Sq^1} w_3w_1 \xrightarrow{Sq^2} w_1w_2w_3.$$

Theorem 4.6. *There is the isomorphism*

$$h^{*,*}(BO(3); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_1, c_2, c_3]\{1, w_1, w_2, Q_0w_2, Q_1w_2, w_3, Q_0w_3, Q_1w_3\} \otimes \mathbb{Z}/2[\tau].$$

where $Q_0w_2 = \tau^{-1}(w_1w_2 + w_3), ..., Q_1w_3 = \tau^{-2}w_1w_2w_3$.

W. S. Wilson ([RWY], [KY]) found a good $Q(i) = \Lambda(Q_0, ..., Q_i)$-module decomposition for $X = BO(n)$, namely,

$$(4.2) \qquad H^*(X; \mathbb{Z}/2) = \bigoplus_{i=-1}^{\infty} Q(i)G_i \quad \text{with} \quad Q_0...Q_iG_i \in t_{\mathbb{C}}(CH^*(X)).$$

Here G_{k-1} is quite complicated; namely, it is generated by symmetric functions

$$\Sigma x_1^{2i_1+1}...x_k^{2i_k+1}x_{k+1}^{2j_1}...x_{k+q}^{2j_q}, \quad k + q \leq n,$$

with $0 \leq i_1 \leq ... \leq i_k$ and $0 \leq j_1 \leq ... \leq j_q$; and if the number of j equal to j_u is odd, then there is some $s \leq k$ such that $2i_s + 2^s < 2j_u < 2i_s + 2^{s+1}$.

Then $w(G_i) \geq i + 1$ in $h^{*,*}(X; \mathbb{Z}/p)$, and so we have

Proposition 4.7. *Letting $w(G_i) = i + 1$, we have the monomorphism*

$$h^{*,*}(BO(n); \mathbb{Z}/2) \subset (\bigoplus_i Q(i)G_i) \otimes \mathbb{Z}/2[\tau].$$

One interesting problem is whether the above injection is really an isomorphism. The similar decomposition holds for $X = (B\mathbb{Z}/p)^n$, and the above injection is an isomorphism. (See Lemma 5.6 below.) The case $X = BO(3)$ is also an isomorphism. Since the direct decomposition of $BO(3) \cong BSO(3) \times B\mathbb{Z}/2$ is complicated, we only write here that of $SO(3)$:

(4.3)

$$\begin{aligned} H^*(BSO(3);\mathbb{Z}/2) &\cong \mathbb{Z}/2[w_2,w_3] \cong \mathbb{Z}/2[c_2,c_3]\{1,w_2,w_3=Q_0w_2,w_2w_3=Q_1w_2\} \\ &\cong \mathbb{Z}/2[c_2,c_3]\{w_2,Q_0w_2,Q_1w_2,c_3=Q_0Q_1w_2\}\oplus \mathbb{Z}/2[c_2] \\ &\cong \mathbb{Z}/2[c_2,c_3]Q(1)\{w_2\}\oplus \mathbb{Z}/2[c_2]. \end{aligned}$$

Since there is the isomorphism $O(2n+1) \cong SO(2n+1) \times \mathbb{Z}/2$, the cohomology of $BSO(2n+1)$ is reduced from that of $BO(2n+1)$. However, the situation for $BO(2n)$ is different. In the next section, we will study $BSO(4)$ for details.

The extraspecial 2-group 2_+^{1+2n} is the n-th central product of the dihedral group D_8 of order 8. It has a central extension

$$0 \to \mathbb{Z}/2 \to G \to V = \bigoplus^{2n} \mathbb{Z}/2 \to 0. \tag{4.4}$$

Let $H^*(BV;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_{2n}]$. Then Quillen proved [Q]

$$H^*(BG;\mathbb{Z}/2) \cong \mathbb{Z}/2[x_1,...,x_{2n}]/(f,Q_0f,...,Q_{n-2}f)\otimes \mathbb{Z}/2[w_{2^n}]. \tag{4.5}$$

Here w_{2^n} is the Stiefel-Whitney class of the real 2^n-dimensional irreducible representation which restricts nonzero on the center, and $f = \sum_i x_{2i-1}x_{2i} \in H^2(BV;\mathbb{Z}/2)$ represents the central extension (4.4).

Letting $y_i = x_i^2$ in $H^*(BG;\mathbb{Z}/2)$, we can write $f^2 = \sum y_{2i-1}y_{2i}$ and

$$\begin{aligned} (Q_{k-1}f)^2 = Q_0Q_kf &= \sum y_{2i-1}^{2^k}y_{2i} - y_{2i-1}y_{2i}^{2^k}, \\ Q_{k-1}f &= \sum y_{2i-1}^{2^{k-1}}x_{2i} - x_{2i-1}y_{2i}^{2^{k-1}}. \end{aligned}$$

Now we consider the motivic cohomology $H^{*,*}(BG;\mathbb{Z}/2)$ and change $y_i = \tau^{-1}x_i^2$. Since $f = 0 \in H^{2,2}(BG;\mathbb{Z}/2)$, we can see that $Q_{k-1}f = 0$ and $Q_kQ_0(f) = 0$ also in $H^{*,*}(BG;\mathbb{Z}/2)$. However, for general n, $\sum y_{2i}y_{2i-1} \neq 0$ in $H^{*,*}(BG;\mathbb{Z}/2)$. Let

$$\begin{aligned} A = (\mathbb{Z}/2[y_1,...,y_{2n},c_{2^n}]/(Q_0Q_kf,...,Q_0Q_nf) \\ \otimes \Delta(x_1,...,x_{2n},w_{2^n})/(f,Q_0f,..,Q_{n-2}f))\otimes \mathbb{Z}/2[\tau]. \end{aligned} \tag{4.6}$$

Lemma 4.8. *For $G = 2_+^{1+2n}$, there is a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection $A/(f^2) \subset h^{*,*}(BG;\mathbb{Z}/2)$.*

When $m = 0,1,-1 \mod 8$ and $m > 0$, we say that $Spin(m)$ is *real type* [Q]. When $Spin(m)$ is real type, from Quillen, we know that $H^*(BSpin(m);\mathbb{Z}/2) \subset H^*(BG;\mathbb{Z}/2)$, where $G = 2_+^{2h+1}$ and h is the Hurwitz number (for details see [Q]).

Corollary 4.9. *Let $G = Spin(m)$ be real type with Hurwitz number h, and let*

$$\begin{aligned} A = (\mathbb{Z}/2[c_2,c_3,,...,c_m,c_{2^h}]/((Q_1Q_0w_2),...,(Q_hQ_0w_2)) \\ \otimes \Delta(w_2,...,w_m,w_{2^h})/(w_2,Q_0w_2,...,Q_{h-2}w_2))\otimes \mathbb{Z}/2[\tau], \end{aligned}$$

where $w_i, i \leq m$ (resp. w_{2^h}) is the Stiefel-Whitney class of the usual $SO(m)$ representation (resp. of the irreducible 2^h-dimensional spin representation). Then we have a map $A \to H^{*,*}(BG;\mathbb{Z}/2)$ which induces the injection

$$A/(c_2) \subset h^{*,*}(BG;\mathbb{Z}/2).$$

We study $Spin(7)$ and the exceptional Lie group G_2. The cohomology of G_2 is given by $H^*(BG_2;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_4,w_6,w_7]$, where w_i is the Stiefel-Whitney class of the inclusion $G_2\subset SO(7)$. The cohomology $H^*(BSpin(7);\mathbb{Z}/2)\cong H^*(BG_2;\mathbb{Z}/2)\otimes \mathbb{Z}/2[w_8]$.

Corollary 4.10. *Let $A=\mathbb{Z}/2[c_2,c_4,c_6,c_7]\otimes\Delta(w_4,w_6,w_7)\otimes\mathbb{Z}/2[\tau]$. Then there is the map $A\to H^{*,*}(BG_2;\mathbb{Z}/2)$ which induces the injection $A/(c_2)\subset h^{*,*}(BG_2;\mathbb{Z}/2)$.*

Remark. Similar facts hold for $BSpin(7)$ tensoring $\mathbb{Z}/2[c_8]$.

The cohomology operations are given by

$$w_4 \xrightarrow{Sq^2} w_6 \xrightarrow{Sq^1} w_7 \xrightarrow{Sq^4} w_4w_7 \xrightarrow{Sq^2} w_7w_6 \xrightarrow{Sq^1} w_7^2,$$
$$Q_1Q_0(w_4w_6)=w_7^2,\quad Q_2Q_1Q_0(w_4w_6w_7)=w_7^4.$$

Proposition 4.11. *Let $w(w_4)=2, w(w_{(4,6)})=2$ and $w(w_{(4,6,7)})=3$ with $t_{\mathbb{C}}(w_{(i_1,...,i_n)})=w_{i_1}...w_{i_n}$. Then $h^{*,*}(BG_2;\mathbb{Z}/2)$ is a subalgebra of*

$$\mathbb{Z}/p[\tau]\otimes\mathbb{Z}/2[c_4,c_6,c_7]\otimes\mathbb{Z}/2\{1,w_4,Sq^2w_4,Q_1w_4,Q_2w_4,Sq^2Q_2w_4,w_{(4,6)},w_{(4,6,7)}\}.$$

Remark. If $t_{\mathbb{C}}^{4,3}\otimes\mathbb{Q}$ is epic, then we can take $w_4\in h^{4,3}(BG;\mathbb{Z}/2)$, *i.e.*, $w(w_4)=2$.

The kernel $\mathrm{Ker}(t_{\mathbb{C}})^{2*,*}$ is not so big for $X=BG_2$. Indeed, it is known [Y3] that

$$CH^*(BG_2)/2\cong\mathbb{Z}/2[c_2,c_4,c_6,c_7]/(rc_2^2,c_2c_7),\quad \text{where } r=0 \text{ or } 1.$$

The cohomology operations are given in $H^*(BSO(7);\mathbb{Z}/2)$ by

$$Q_1Q_0w_2=w_3^2,\quad Q_2Q_0w_2=w_5^2,\quad Q_3Q_0w_2=w_7^2w_2^2+w_6^2w_3^2+w_5^2w_4^2.$$

Hence we have $c_3=0, c_5=0$ and $c_2c_7=0$ in $CH^*(BG_2)/2$, but $c_2\neq 0$.

From here we consider the case $p=odd$. One of the easiest examples is the case $G=PGL_3$ and $p=3$. The mod 3 cohomology is given by ([KY], [Ve1])

$$(\mathbb{Z}/3[y_2]\{y^2\}\oplus\mathbb{Z}/3\{1,y_2,y_3,y_7\}[y_8])\otimes\mathbb{Z}/3[y_{12}]$$

It is known that y_2^2,y_2^3,y_8^2 and y_{12} are represented by Chern classes. Moreover, $Q_1Q_0(y_2)=y_8$. Hence these elements are in the Chow ring; namely,

$$h^{2*,*}(BPGL_3;\mathbb{Z}/3)\cong(\mathbb{Z}/3[y_2]\{y_2^2\}\oplus\mathbb{Z}/3[y_8])\otimes\mathbb{Z}/3[y_{12}].$$

The cohomology operations are given by

$$y_2 \xrightarrow{\beta} y_3 \xrightarrow{P^1} y_7 \xrightarrow{\beta} y_8. \tag{4.7}$$

Thus we get $h^{*,*}(PGL_3;\mathbb{Z}/3)$ completely.

Theorem 4.12. *Letting $w(y_2)=2$, we have the isomorphism*

$$h^{*,*}(BPGL_3;\mathbb{Z}/3)\cong(\mathbb{Z}/3[y_2]\{y^2\}\oplus\mathbb{Z}/3\{1\}\oplus\mathbb{Z}/3[y_8]\otimes Q(1)\{y_2\})\otimes\mathbb{Z}/3[y_{12},\tau].$$

Next consider the extraspecial p-group $G=p_+^{1+2n}$. When $n>2$, even the cohomology rings $H^*(BG;\mathbb{Z}/p)$ are unknown, while it contains the subring [TeY1]

$$R=\mathbb{Z}/p[y_1,...,y_{2n},c_{p^n}]/(Q_1Q_0f,...,Q_nQ_0f), \tag{4.8}$$

where $f=\sum^n x_{2i-1}x_{2i}$ for $\beta x_i=y_i$ and $Q_kQ_0f=\sum y_{2i-1}y_{2i}^{p^k}-y_{2i-1}^{p^k}y_{2i}$. Since $f=0\in H^{2,2}(BG;\mathbb{Z}/p)$, we have

Proposition 4.13. *There is the injection*

$$R\otimes\mathbb{Z}/p[\tau]\hookrightarrow H^{*,*}(Bp_+^{1+2n};\mathbb{Z}/p).$$

We consider here other arguments for a different but similar group. Let $\tilde{p}_+^{1+2n}$ be the central product of p_+^{1+2n} and the circle, i.e. $\tilde{p}_+^{1+2n} = p_+^{1+2n} \times_C S^1$, identifying $C \cong \mathbb{Z}/p \subset S^1$, where C is the center. Let us write

$$e_A = \prod_{0 \neq (\lambda_1, \lambda_3, ..., \lambda_{2n-1})} (\lambda_1 y_1 + \cdots + \lambda_{2n-1} y_{2n-1}). \tag{4.9}$$

If we localize by inverting e_A, then the cohomology of $\tilde{p}_+^{1+2n}$ is expressed easily [Y2] as

$$[e_A^{-1}]H^*(B\tilde{p}_+^{1+2n}; \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \Lambda(x_1, x_3, ..., x_{2n-1}), \qquad \beta(x_i) = y_i. \tag{4.10}$$

Theorem 4.14. *Letting $w(x_i) = 1$, we have the ring isomorphism*

$$[e_A^{-1}]h^{*,*}(B(\tilde{p}_+^{1+2n}); \mathbb{Z}/p) \cong [e_A^{-1}]R \otimes \mathbb{Z}/p[\tau] \otimes \Lambda(x_1, x_3, \ldots, x_{2n-1}).$$

Proof. There is the splitting abelian subgroup $(\mathbb{Z}/p)^n \cong A \subset \tilde{p}_+^{1+2n}$ such that

$$h^{*,*}(BA; \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau, y_1, y_3, ..., y_{2n-1}] \otimes \Lambda(x_1, x_3, ..., x_{2n-1}).$$

Each monomial $x_{i_1}...x_{i_s}$, $1 \leq i_1, ..., i_s \leq 2n-1$, is a $\mathbb{Z}/p[\tau]$-module generator in the above cohomology, hence also in the cohomology of $B\tilde{p}_+^{1+2n}$. □

We consider the case $n = 1$ here. Let us write $E = p_+^{1+2}$ for each odd prime p. The ordinary cohomology is known by Lewis [Ly], [TeY2]; namely,

$$H^{even}(BE)/p \cong (\mathbb{Z}/p[y_1, y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus \mathbb{Z}/p\{c_2, ..., c_{p-1}\}) \otimes \mathbb{Z}/p[c_p],$$
$$H^{odd}(BE) \cong \mathbb{Z}/p[y_1, y_2, c_p]\{a_1, a_2\}/(y_1 a_2 - y_2 a_1, y_1^p a_2 - y_2^p a_1), \quad |a_i| = 3.$$

It is also known that $Q_1(a_i) = y_i c_p$ and $order(c_p) = p^2$.

The group 2_+^{1+2} is the dihedral group D_8 of order 8. The integral cohomologies are

$$H^{even}(BD_8)/2 \cong \mathbb{Z}/2[y_1, y_2, c_2]/(y_1 y_2), \quad H^{odd}(BD_8) \cong H^{even}(BD_8)/2\{e\}$$

where $c_2 = w_2^2$, $e = (x_1 + x_2)w_2$ in $H^*(BD_8; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, w_2]/(x_1 x_2)$ and $Q_1 e = (y_1 + y_2)c_2$, $order(c_2) = 4$.

Theorem 4.15. *For all primes p, we have the isomorphisms*

$$h^{*,*}(Bp_+^{1+2}; \mathbb{Z}/p) \cong (\{1, \partial_p^{-1}\}(H^*(Bp_+^{1+2})/p) - \{\partial_p^{-1}1\}) \otimes \mathbb{Z}/p[\tau],$$

where $w(H^{even}(Bp_+^{1+2})/p) = 0, w(H^{odd}(Bp_+^{1+2})) = 1$ and $w(\partial_p^{-1}(x)) = w(x) + 1$.

Proof. We will prove this only for odd primes, since the proof for $p = 2$ is similar. Since all elements in $H^{even}(BE)$ are generated by Chern classes, we have the isomorphism $h^{2*,*}(BG; \mathbb{Z}/p) \cong H^{even}(BE)/p$. We know $H^{odd}(BE; \mathbb{Z}/p)$ is generated as an $H^{even}(BE)/p$-module by two elements a_1, a_2 such that $Q_1 a_i = y_i c_p$ [TeY2].

The mod p cohomology is written additively, $H^*(BE; \mathbb{Z}/p) \cong \{1, \partial_p^{-1}\}H^*(BE)/p$. Here ∂_p is the (higher) Bockstein operator. All elements in $H^{odd}(BE)$ are just p-torsion, and we can take $a_i' \in H^2(BE; \mathbb{Z}/p)$ such that $\beta(a_i') = a_i$. Thus we take $a_i' \in H^{2,2}(BE; \mathbb{Z}/p)$ so that $a_i \in H^{3,2}(BE; \mathbb{Z}/p)$.

Next consider elements $x = \partial_p^{-1}(y)$, $y \in H^{even}(BE)/p$. If $y \in (\mathrm{Ideal}(y_1, y_2))$, then $\partial_p^{-1}(y) = \sum x_i b_i$ for $b_i \in H^{even}(BE)/p$, and hence we can take $w(\partial_p^{-1}(y)) = 1$. For other elements $y = c_i c_p^n$, $2 \leq i \leq p-1$, it is known [Ly] that $c_i = \mathrm{Cor}_M^E(u^i)$

with $0 \neq u \in H^2(B\mathbb{Z}/p;\mathbb{Z}/p)$ for a maximal abelian subgroup $M \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Hence $y \in H^{2*,*}(BE;\mathbb{Z}/p)$ is also p-torsion. Considering the exact sequence

$$\to H^{2*-1,*}(BE;\mathbb{Z}/p^N) \to H^{2*,*}(BE) \xrightarrow{p^N} H^{2*,*}(BE) \to,$$

we get $w(\partial_p^{-1}(y)) = 1$. The element $y = c_p^n$ is p^2-torsion in $H^*(BE;\mathbb{Z}/p)$. Note that $\mathrm{Cor}_M^E(u^{pn}) = pc_p^n + k$ with $k \in \mathrm{Ideal}(y_1, y_2)$. Thus $y \in H^{2*,*}(BE;\mathbb{Z}/p)$ is also p^2-torsion. Then we can take $w(\partial_p^{-1}(y)) = 1$. This completes the proof. $\square$

We easily compute the following results.

Corollary 4.16. *For each prime p, there is an isomorphism*

$$h^{*,*}(Bp_+^{1+2};\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes (\mathbb{Z}/p\{1\} \oplus Q'(0)G_0' \oplus Q(0)G_0 \oplus Q(1)G_1),$$

where $Q'(0) = \Lambda(\beta_{p^2})$, β_{p^2} *is the* p^2*-torsion Bockstein operator, and if* $p = 2$, *then*

$$\begin{cases} G_0' \cong \mathbb{Z}/2[c_2]\{x_1w_2\}, \quad \beta_4(x_1w_2) = c_2, \\ G_0 \cong \mathbb{Z}/2[y_1]\{x_1\} \oplus \mathbb{Z}/2[y_2]\{x_2\} \oplus \mathbb{Z}/2[c_2]\{x_1c_2\}, \\ G_1 \cong \mathbb{Z}/2[y_1,y_2,c_2]/(y_1y_2)\{w_2\}, \end{cases}$$

and if p is an odd prime, then

$$\begin{cases} G_0' \cong \mathbb{Z}/p[c_p]\{c_p'\}, \quad \beta_{p^2}(c_p') = c_p, \\ G_0 \cong \mathbb{Z}/p[y_1,y_2]\{x_1,x_2\}/(y_2x_1 - y_1x_2, y_2^px_1 - y_1^px_2) \oplus \mathbb{Z}/p[c_p]\{c_2',...,c_{p-1}'\}, \\ G_1 \cong \mathbb{Z}/p[y_1,y_2,c_p]\{a_1',a_2'\}/(y_2a_1' - y_1a_2', y_2^pa_1' - y_1^pa_2'), \quad \beta(a_i') = a_i,\ \beta(c_i') = c_i. \end{cases}$$

5. BP-theory and $\mathrm{Ker}\, t_{\mathbb{C}}^{2*,*}$

In this section, we always assume $k = \mathbb{C}$. Even this case it seems difficult to know $\mathrm{Ker}\, t_{\mathbb{C}}$. For Chow rings $CH^*(X)$, Totaro found a good way to get nonzero elements in $\mathrm{Ker}\, t_{\mathbb{C}}$. Let $MU^*(-)$ (resp. $BP^*(-)$) be the complex cobordism theory (resp. Brown-Peterson theory) with the coefficient ring $MU^* = MU^*(pt) = \mathbb{Z}[x_1,...]$, $|x_i| = -2i$ (resp. $BP^* = \mathbb{Z}_{(p)}[v_1,...]$, $|v_i| = -2(p^i-1)$). The Thom map induces $\rho : MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \to H^*(X(\mathbb{C});\mathbb{Z})$. Totaro constructed [To1] the map

$$\tilde{cl} : CH^*(X) \to MU^*(X(\mathbb{C})) \otimes_{MU^*} \mathbb{Z} \tag{5.1}$$

such that the composition $\rho\tilde{cl}$ is the usual cycle map $cl = t_{\mathbb{C}}^{2*,*}$, which is also the realization map.

In this section, hereafter, X is just a topological space, e.g., $X(\mathbb{C})$, to simplify the notation. Since $BP^*(X) \cong BP^* \otimes_{MU^*_{(p)}} MU^*(X)_{(p)}$, the similar fact holds for BP-theory. Let $P(n)^* = BP^*/(p, v_1, ..., v_{n-1})$, e.g., $P(0)^* = BP^*$, $P(1)^* = BP^*/p$ and $P(\infty)^* = \mathbb{Z}/p$. Then there are cohomology theories $P(n)^*(-)$ with the coefficient $P(n)^*(pt) \cong P(n)^*$, e.g., $P(0)^*(X) = BP^*(X)$, $P(1)^*(X) = BP^*(X;\mathbb{Z}/p)$ and $P(\infty)^*(X) = H^*(X;\mathbb{Z}/p)$. Hence there are maps of cohomology theories

$$\begin{aligned} cl_p : CH^*(-)/p \to BP^*(-) \otimes_{BP^*} \mathbb{Z}/p \to ... \to P(n)^*(-) \otimes_{P(n)^*} \mathbb{Z}/p \\ \to P(n+1)^*(-) \otimes_{P(n+1)^*} \mathbb{Z}/p \to ... \to H^*(-;\mathbb{Z}/p) \end{aligned}$$

such that the composition is the cycle map $cl_p = t_{\mathbb{C}}$. The Morava K-theory is defined by $K(n)^*(X) = P(n)^*(X) \otimes_{P(n)^*} K(n)^*$, where $K(n)^* = \mathbb{Z}/p[v_n, v_n^{-1}]$. In general, $K(n)^*(X) \not\cong K(n)^* \otimes_{BP^*} BP^*(X)$. However, when $K(n)^{odd}(X) = 0$, it is known [RWY] that

$$P(n)^*(X) \cong BP^*(X) \otimes_{BP^*} P(n)^*, \quad K(n)^*(X) \cong BP^*(X) \otimes_{BP^*} K(n)^*.$$

We know that $K(n)^{odd}(BG) = 0$ for many cases, while Kříž showed $K(n)^*(BG') \neq 0$ for some fine group G'.

One useful tool for computing $BP^*(X)$ is the Atiyah-Hirzebruch spectral sequence [TeY2], [KY]

$$E_2^{*,*} = H^*(X) \otimes BP^* \Longrightarrow BP^*(X).$$

It is known that $d_{2p^i-1}(x) = v_i \otimes Q_i(x) \operatorname{mod}(M_i)$, where M_i is the ideal of $E_{2p^i-1}^{*,*}$ generated by elements in $(p, v_1, ..., v_{i-1})E_2^{*,*}$. Here we assume that $H^*(X)$ has no higher p-torsion and that

(5.2) All nonzero differentials are of the form

$$d_{2p^i-1}(x) = v_i \otimes Q_i(x) \operatorname{mod}(M_i).$$

Let us write

$$grBP^*(X) \cong E_\infty^{*,*} \cong A \oplus B \tag{5.3}$$

where A (resp. B) is a BP^*-module generated by nonzero elements in $H^*(X)/p$ (resp. $pH^*(X) \oplus E_\infty^{*,minus}$), so that $B \subset \operatorname{Ker}(\rho_p)$. We can write

$$A \cong \bigoplus_{n=-1}^{\infty} P(n+1)^* \tilde{G}_n$$

by the prime invariant ideal theorem of Landweber; if $P(n)^*/(a)$ is a $BP^*(BP)$-module, then $a = v_n^s$ for some $s \geq 1$.

Take a nonzero element $\tilde{g}_n \in \tilde{G}_n$ for $n \geq 2$. Since $\tilde{g}_n$ is $(p, ..., v_n)$-torsion, there is $g_{(n,s)} \in E_{2p^s-1}^{*,0}$ such that $d_{2p^s-1}(g_{(n,s)}) = v_s \otimes \tilde{g}_n$ for each $1 \leq s \leq n$. Let the BP^*-module in $E_{2p^s-1}^{*,*}$ generated by $g_{(n,s)}$ be isomorphic to a $P(s'+1)^*$-free module for $s' < s$. Here note that if $s' \neq s-1$, then $\operatorname{Ideal}(v_{s'+1}, .., v_{s-1})\{g_{n,s}\} \subset \operatorname{Ker}(d_{2p^s-1})$. In any case, we can take $g_{(n,s,t)} \in H^*(X)/p$ for $t < s'$ such that $d_{2p^t-1}(g_{(n,s,t)}) = v_t \otimes g_{(n,s)}$. Continuing this argument we can take

$$\tilde{g}_n \xleftarrow{Q_{s_1}} g_{(n,s_1)} \xleftarrow{Q_{s_2}} g_{(n,s_1,s_2)} \longleftarrow \ ... \ \xleftarrow{Q_{s_m}} g_{(n,s_1,...,s_m)}$$

for some $(n > s_1 > ... > s_m)$.

Lemma 5.1. *Let $H^*(X)_{(p)}$ have no higher p-torsion. Suppose (5.2) holds, and $A = \bigoplus_{n=-1} P(n+1)^* \tilde{G}_n$ in (5.3). Then there is the injection*

$$H^*(X; \mathbb{Z}/p) \hookrightarrow \bigoplus_n Q(n) G_n \quad \text{with } Q_0 ... Q_n G_n = \tilde{G}_n.$$

Proof. Let H be a $\mathbb{Z}/p$-module generated by elements $g_{(n,s_1,...,s_m)}$ discussed above. Define the map $j_{\mathbb{C}} : H \hookrightarrow \oplus Q(n) G_n$ by

$$j_{\mathbb{C}}(g_{(n,s_1,...,s_m)}) = Q_{s_m}^{-1} ... Q_{s_1}^{-1}(\tilde{g}_n) = Q_0 ... \hat{Q}_{s_m} .... \hat{Q}_{s_1} ... Q_n(g_n), \quad Q_0 ... Q_n g_n = \tilde{g}_n.$$

Suppose $x \in H^*(X)_{(p)} - H$. Then by the assumption (5.3), x is not a permanent cycle. Hence $d_{2p^i-1}(x) \neq 0$ for some i, and so $Q_i(x) \neq 0$. Let t be a largest number such that $Q_{i_t} ... Q_{i_1} Q_i x = \tilde{g} \neq 0$. Since $Q_j(\tilde{g}) = 0$ for all j, we know $\tilde{g}$ is a permanent cycle. This element $\tilde{g} \in E_\infty^{*,0}$ generates a $P(N+1)^*$-module for $N = \max(i_s, ..., i_1, i)$. This means $x = (Q_i^{-1} Q_{i_1}^{-1} ... Q_{i_s}^{-1} \tilde{g}) \in H$. □

Let us write $Q(i,n) = \Lambda(Q_i, ..., Q_n)$, so that $Q(0,n) = Q(n)$.

Lemma 5.2. *Let $H^*(X)_{(p)}$ have no higher p-torsion.*

(1) *If (5.2) is satisfied and, in (5.3),*

$$A = \bigoplus_{n=-1} P(n+1)^*\tilde{G}_n \quad \text{and } B \cong \bigoplus_{s=0} BP^*\{p, v_1, .., v_s\}\tilde{K}_s,$$

then we have the isomorphisms

$$H^*(X)/p \cong (\tilde{G}_{-1} \oplus \tilde{G}_0 \oplus \bigoplus_{n=1} Q(1,n)G'_n - \bigoplus_{s=0}(Q(1,s)K'_s - \tilde{K}'_s)),$$

$$H^*(X;\mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(n)G_n - \bigoplus_{s=0}(Q(s)K_s - \tilde{K}_s))$$

with $Q_0...Q_nG_n = \tilde{G}_n$, $Q_0G_n = G'_n$ and $Q_0...Q_sK_s = \tilde{K}_s$, $Q_0K_s = K'_s$.

(2) *If $Q_0...Q_nG_n \in \mathrm{Im}(\rho)$ and the degrees of $\tilde{K}_s$ and $\tilde{G}_n$ are even, then the converse also holds.*

Proof. (1) Let $0 \neq x \in \tilde{K}_s$. Since x is not a permanent cycle, $d_{2p^i-1}(x) \neq 0$ and $Q_i(x) \neq 0$. Since $\{p, ..., v_s\}\tilde{K}_s$ are permanent cycles, we know $Q_i(x) \in E^{*,*}_{2p^i-1}$ is a $P(s+1)^*$-module, that is, $i = s+1$ by the Landweber invariant prime ideal theorem, and

$$\bigoplus Q(n)G_n \supset Q(s)K_s.$$

Since v_ix generates a free BP^*-module, $x \notin \mathrm{Im}(Q_j)$ for all j. Hence we get the injection

$$H^*(X;\mathbb{Z}/p) \hookrightarrow \bigoplus Q(n)G_n - (Q(s)K_s - \tilde{K}_s).$$

Let $x = Q_{i_1}...Q_{i_k}g_n$ be in the right-hand side of the above injection, and such that $0 \neq Q_i(x) \in H^*(X;\mathbb{Z}/p)$ but $x \notin H^*(X;\mathbb{Z}/p)$. If $Q_i(x)$ is not a permanent cycle, then $v_iQ_i(x)$ is permanent, so $Q_i(x)$ must be in $\tilde{K}_s$ and hence $x \in Q(s)K_s$; this is a contradiction. Otherwise $Q_i(x) = \tilde{g}_n$ generates a $P(n)^*$-module and $Q_i(x)$ must be $\mathrm{Im}(Q_j)$ for all $j \leq n$. Hence $x \in H^*(X;\mathbb{Z}/p)$.

(2) By induction on i, we assume $E^{*,*}_{2p^{i-1}} \cong C(i) \oplus D(i)$, where

$$C(i) = P(i)^*(\bigoplus_{i\leq n} Q(i,n)Q_{i-1}...Q_0G_n - \bigoplus_{i\leq s} Q(i,n)Q_{i-1}...Q_0K_s) \oplus \bigoplus_{i-1\leq s} BP^*\tilde{K}_s,$$

$$D(i) = \bigoplus_{n\leq i-1} P(n+1)^*\tilde{G}_n \oplus \bigoplus_{s\leq i-2} BP^*\{p, ..., v_s\}\tilde{K}_s.$$

Here elements of $\tilde{K}_s$ and $D(i)$ are even dimensional. Hence all odd dimensional elements generate free $P(i)^*$-modules. Note that if $i > j$, then there are no non-trivial maps from $P(i)^*$-modules to free $P(j)^*$-modules. We also note that there is no possibility that $d_t(v_kx) = v_iy$ for $x \in \tilde{K}_s$ and $y \in E_t^{odd,*}$, $t < 2p^j - 1$. Indeed there is the map i^* of spectral sequences from that for $BP^*(X)$ to that for $P(i)^*(X)$; in the last spectral sequence $E^{*,*}_{2p^i-1} \cong P(i)^* \otimes H^*(X;\mathbb{Z}/p)$ and $i^*(v_iy) \neq 0$. Hence the next nonzero differential must be of the form $d_{2p^i-1}(x) = v_i \otimes Q_i(x)$. Therefore we have

$$E^{*,*}_{2p^i} \cong C(i+1) \oplus D(i) \oplus P(i+1)Q_i...Q_0G_i \oplus BP^*\{p, ..., v_{i-1}\}\tilde{K}_{i-1}.$$

The last term is computed from $Q_i\tilde{K}_{i-1} \neq 0$ and $\mathrm{Ker}\, d_{2p^i-1}|BP^*\{\tilde{K}_{i-1}\} = BP^*\{p, ..., v_{i-1}\}\tilde{K}_{i-1}$, since $Q_i\tilde{K}_{i-1}$ is $P(i)^*$-free in $E^{*,*}_{2p^i-1}$. □

The classifying spaces of groups $BO(n), SO(4), G_2, Spin(m), m \leq 9$ for $p = 2$ and PGL_3, F_4 for $p = 3$, and $(\mathbb{Z}/p)^n$ satisfy the assumption of the above lemma. However $SO(6)$ does not satisfy the above lemma [I].

We will show that the isomorphism (1) in Lemma 5.2 approximates $h^{*,*}(X;\mathbb{Z}/p)$. Let $Ih^{*,*}(X)$ be a $\mathbb{Z}/p[\tau]$-submodule of $h^{*,*}(X;\mathbb{Z}/p)$ generated by image from $h^{*,*}(X)/p$. The following theorem is almost immediate.

Theorem 5.3. *Suppose that (1) in Lemma 5.2 holds. Then we have the injection*

$$Ih^{*,*} \hookrightarrow ((G_{-1}/p \bigoplus_{n=1} Q(1,n)G'_n) - (\bigoplus_{s=1} Q(1,s)K'_s - \tilde{K}'_s)) \otimes \mathbb{Z}/p[\tau],$$

$$h^{*,*}(X;\mathbb{Z}/p) \hookrightarrow (\bigoplus_{n=1} Q(n)G_n - (\bigoplus_{s=1} Q(s)K_s - \tilde{K}_s)) \otimes \mathbb{Z}/p[\tau],$$

with $w(G_n) = n+1$, $w(G'_n) = n$. Moreover, if some BP^*-module generator in $\mathrm{Ideal}(p,..,v_1)\tilde{K}_s \subset E_\infty^{*,*}$ *is represented by transfer of a Chern class, then* $\mathrm{Ker}(t_{\mathbb{C}}^{2*,*})$ *contains a nonzero element.*

The $P(m)^*(-)$ version of above facts also holds, if we consider the spectral sequence

$$E_2^{*,*} = H^*(X;\mathbb{Z}/p) \otimes P(m)^* \Longrightarrow P(m)^*(X).$$

(5.3)$'$ Let $E_\infty^{*,*} = A \oplus B$, where A (resp. B) is the $P(m)^*$-module generated by generators in $E_\infty^{*,0}$ (resp. in $E_\infty^{*,minus}$).

Lemma 5.4. (1) *If (5.2) is satisfied and, in (5.3)$'$,*

$$A \cong \bigoplus_{n=-1} P(m+n+1)^*\tilde{G}_n(m), \quad B \cong \bigoplus_{s=0} P(m)^*\{v_m, ..., v_s\}K_s(m),$$

then we have the isomorphism

$$H^*(X;\mathbb{Z}/p) \cong (\bigoplus_{n=-1} Q(m,n+m)G_n(m)) - \bigoplus_{s=0} Q(m,m+s)K_s(m) - \tilde{K}_s(m))$$

with $Q_m...Q_{m+n}G_n(m) = \tilde{G}_n(m)$ and $Q_m...Q_{m+s}K_s(m) = \tilde{K}_s(m)$.

(2) *If $Q_m...Q_{m+n}G_n(m) \in \mathrm{Im}(\rho)$ and $|\tilde{K}_s(m)| = even$, then the converse also holds.*

The $P(m)^*$-versions also hold for $G = (\mathbb{Z}/p)^n, BO(n), BSO(4), p_+^{1+2}$. One application for the above lemma is the following.

Corollary 5.5. *Let $H^*(X;\mathbb{Z}/p)$ (resp. $H^*(Y;\mathbb{Z}/p)$) have the decomposition of Lemma 5.2 (1) (resp. Lemma 5.4 (1) for all $m \geq 0$). Then $H^*(X \times Y;\mathbb{Z}/p)$ also has decomposition similar to that of Lemma 5.2 (1).*

Proof. We get the following isomorphism:

$$Q(n-1)G_{n-1} \otimes H^*(Y;\mathbb{Z}/p)$$

$$\cong Q(n-1)G_{n-1} \otimes (Q(n,n+k)G_k(n) - \bigoplus Q(n,n+t)K_t(n) - \tilde{K}_t(n)))$$

$$\cong (Q(n+k)G_{n-1} \otimes G_k(n)) - (Q(n+t)G_{n-1} \otimes K_t(n) - Q(n-1)G_{n-1} \otimes \tilde{K}_s(n)),$$

since each Q_i is derivative. □

Lemma 5.6. *If $H^*(X;\mathbb{Z}/p) \cong \bigoplus Q(n)G_n$, then $H^*(X \times B\mathbb{Z}/p) \cong \bigoplus Q(n)G'_n$, where*

$$G'_n \cong \mathbb{Z}/p[y]/(y^{p^n})G_n \oplus \mathbb{Z}/p[y]G_{n-1}\{x\}.$$

Proof. Since we have the decomposition

$$H^*(B\mathbb{Z}/p;\mathbb{Z}/p)\cong \mathbb{Z}/p[y]/(y^{p^n})\oplus \mathbb{Z}/p[y]Q(n,n)\{x\},$$

we get the lemma. □

When $X=(B\mathbb{Z}/p)^n$, inductively we get the decomposition $H^*((B\mathbb{Z}/p)^n;\mathbb{Z}/p)\cong \bigoplus Q(n)G_n$. Hence $B=0$ and

$$grBP^*(X)\cong \bigoplus P(n+1)^*\tilde{G}_n,\quad H^{*,*}(X;\mathbb{Z}/p)\cong \bigoplus Q(n)G_n\otimes \mathbb{Z}/p[\tau].$$

Of course these are given in (3.9). The similar facts also hold for $X=BO(n)$. Moreover, W. S. Wilson proved [RWY] that

$$BP^*(BO(n))\cong BP^*[c_1,...,c_n]/(c_1-c_1^*,...,c_n-c_n^*),$$

where c_i^* is the complex conjugate of the Chern class of the usual complex representation. The cohomology $h^{*,*}(BO(n))$ is studied in (4.2).

Next consider the case $X=BSO(4)$. The mod 2 cohomology is $H^*(X;\mathbb{Z}/2)\cong \mathbb{Z}/2[w_2,w_3,w_4]$. The cohomology operation acts as

$$Q_0w_2=w_3,\ Q_1w_3=w_3^2,\ Q_1w_4=w_4w_3,\ Q_1Q_2w_4=w_3^2w_4^2.$$

The integral cohomology is written as

$$H^*(X)_{(2)}\cong Z_{(2)}[w_2^2,w_4]\otimes (Z_{(2)}\{1\}\oplus \mathbb{Z}/2[w_3]\{w_3\}).$$

In the Atiyah-Hirzebruch spectral sequence, nonzero differentials are $d_{2^{i+1}-1}(x)=v_i\otimes Q_i(x)$ for $i=1,2$. We can compute

$$E_\infty^{*,*}\cong E_8^{*,*}\cong Z_{(2)}[c_2]\otimes (BP^*[c_4]\{1,2w_4\}\oplus P(2)^*[c_3]\{c_3\}\oplus P(3)^*[c_3,c_4]\{c_3c_4\}),$$
$$BP^*(X)\otimes_{BP^*}Z_{(2)}\cong Z_{(2)}[c_2,c_4]\otimes (Z_{(2)}\{1,2w_4\}\oplus \mathbb{Z}/2[c_3]\{c_3\}).$$

Hence the assumption of (1) in Lemma 5.2 is satisfied by

$$\tilde{G}'_{-1}\cong \mathbb{Z}/2[c_2,c_4],\quad \tilde{G}'_1=\mathbb{Z}/2[c_2,c_3]\{c_3\}\quad \tilde{G}'_2=\mathbb{Z}/2[c_2,c_3,c_4]\{c_3c_4\},$$
$$\tilde{K}'_0=\mathbb{Z}/2[c_2,c_4]\{2w_4\}.$$

Therefore we get

Proposition 5.7. *Let $w(w_4)=2$. Then the bidegree $\mathbb{Z}/2[\tau]$-module $Ih^{*,*}(BSO(4))$ (resp. $h^{*,*}(BSO(4);\mathbb{Z}/2)$) is isomorphic to a bidegree $\mathbb{Z}/2[\tau]$- submodule of*

$$\mathbb{Z}/2[\tau,c_2]\otimes (\mathbb{Z}/2[c_4]\{1\}\oplus \mathbb{Z}/2[c_3]\otimes Q(1,1)\{w_3\}\oplus \mathbb{Z}/2[c_3,c_4]\otimes Q(1,2)\{w_4\}))$$

(resp. $\mathbb{Z}/2[\tau,c_2]\otimes(\mathbb{Z}/2[c_4]\{1\}\oplus\mathbb{Z}/2[c_3]\otimes Q(1)\{w_2\}\oplus\mathbb{Z}/2[c_3,c_4]\otimes(Q(2)-\mathbb{Z}/p)\{a\})$,

where $Q_0a=w_4$.

Remark. If $w_4\in H^{4,3}(BSO(4))$, then $Ih^{*,*}(BSO(4))$ is isomorphic to the $\mathbb{Z}/2[\tau]$-module in the above proposition.

Remark. For this case, we have $K_0=\mathbb{Z}/2[c_2]\{a\}$ and $Q_0K_0={K'}_0$ in Lemma 5.2. Indeed, $Q_0a=w_4$. However, $w_4\notin \mathrm{Im}(Q_0)$ in $h^{*,*}(BSO(4);\mathbb{Z}/2)$, because a itself does not exist in $h^{*,*}(BSO(4);\mathbb{Z}/2)$.

We know that the element corresponding to $2w_4$ is represented by a Chern class c_2' of some representation, and this means the Totaro's cycle map $\tilde{cl}$ is epic. Indeed, Totaro and Pandharipande showed that this map is isomorphic, namely,

$$CH^*(BSO(4))_{(2)} \cong Z_{(2)}[c_2, c_3, c_4, c_2']/(2c_3, c_3c_2', {c_2'}^2 - 4c_4).$$

Next consider the $P(1)^*$-version for $BSO(4)$. By using the computations of Q_iw_j [I] and the Atiyah-Hirzebruch spectral sequence, we can prove that

$$grP(1)^*(BSO(4)) \cong P(1)^*[c_4]\{1, v_1w_2w_4\} \oplus P(2)^*\{c_3\}$$
$$\oplus P(3)^*[c_3]\{c_3^2, c_3c_4\} \oplus P(3)^*[c_4]\{c_3c_4^2\} \oplus P(4)^*[c_3, c_4]\{c_3^2c_4^2\}$$

We have another decomposition of $H^*(BSO(4); \mathbb{Z}/2)$.

Proposition 5.8.

$$H^*(BSO(4); \mathbb{Z}/2) \cong \mathbb{Z}/2[c_4] \oplus Q(1, 1)\{w_3\} \oplus \mathbb{Z}/2[c_3] \otimes (Q(1, 2)\{w_2, w_4\})$$
$$\oplus \mathbb{Z}/2[c_4] \otimes (Q(1, 2)\{c_4w_4\}) \oplus \mathbb{Z}/2[c_3, c_4] \otimes (Q(1, 3)\{Q_1^{-1}w_2w_4\} - \{Q_1^{-1}w_2w_4\}).$$

We consider the relation between $grBP^*(X)$ and $grP(1)^*(X)$. When $X = BSO(4)$, it is known [KY] that $K(n)^{odd}(X) = 0$, and hence

$$P(m)^*(X) \cong P(m)^* \otimes_{BP^*} BP^*(X).$$

Therefore no $P(m)^*(X)$ is v_m-torsion. Of course we have already seen that for the $grBP^*(-)$-versions the above facts do not hold. If there is a relation $pa_0 + v_1a_1 + v_2a_2 + ... = 0 \in BP^*(X)$, then it is known [Y1] that there is $y \in H^*(X; \mathbb{Z}/p)$ such that $Q_i(y) = \rho(a_i)$, where $\rho : BP^*(X) \to H^*(X; \mathbb{Z}/p)$ is the Thom map. In $H^*(BSO(4); \mathbb{Z}/2)$, we see that

$$Q_0(w_2w_3) = c_3,\ Q_1(w_2w_3) = 0,\ Q_2(w_2w_3) = c_3^2.$$

Hence we have the relation $2c_3 + v_2c_3^2 + ... = 0 \in BP^*(BSO(2))$. This shows that c_3^2 is $P(2)^*$-free in $grBP^*(BSO(4))$, but it is a $P(3)^*$-free module in

$$grP(1)^*(BSO(4)) = gr(BP^*(BSO(4))/2).$$

We also see that for $x = c_3w_3w_4 + c_4w_2w_3$

$$Q_0(x) = c_3c_4, \quad Q_1(x) = Q_2(x) = 0, \quad Q_3(x) = c_3^2c_4^2.$$

This means that $2c_3c_4 + v_3c_3^2c_4^2 + ... = 0 \in BP^*(BSO(4))$. Hence $c_3^2c_4^2$ is a $P(3)^*$-free module in $grBP^*(BSO(4))$ but is a $P(4)^*$-free module in $gr(BP^*(BSO(4))/2)$.

Next consider the case $X = BSO(6)$. In this case the assumption (5.3) is not satisfied. In fact, Inoue computed [I]

$$grBP^*(BSO(6)) \cong \bigoplus_{n=-1}^{4} P(n+1)^*\tilde{G}_n \oplus P(2)^*/(v_2^2)\tilde{G}_1' \oplus BP^*\{2\}\tilde{K}_0.$$

(For details, see [I].) In particular, he showed that

$$d_5(2w_6) = v_1^2w_6w_5, \quad d_{11}(v_1 \otimes w_6w_5) = v_2^2w_6^2w_5^2.$$

However, even this case we can show that

$$H^*(BSO(6); \mathbb{Z}/2) \subset \bigoplus Q(n)G_n \oplus Q(1)G_1'.$$

Moreover, R. Field [F] announced that

$$CH^*(BSO(2n)) \cong Z_{(2)}[c_2, ..., c_{2n}, y_n]/(2c_{odd}, c_{odd}y_n, y_n^2 - (-1)^n 2^{2n-2}c_{2n})$$

with $deg(y_n) = 2n$. Hence Ideal$(y_n) \subset$ Ker$(t_{\mathbb{C}})$. However, y_n is not represented by a Chern class of any representation for $n > 2$. We also note that $BP^*(BSO(2n))$ are not known for $n > 3$.

The cases $X = BG_2, BSpin(7)$ are quite similar to the case $X = BSO(4)$. Indeed, $CH^*(BG_2)/2$ and $h^{*,*}(BG_2; \mathbb{Z}/2)$ have been discussed in §4, and

$$grBP^*(BG_2) \cong Z_{(2)}[c_4, c_6] \otimes (BP^*\{1, 2w_4\} \oplus P(3)^*[c_7]\{c_7\}).$$

The infinite term of the spectral sequence for $BP^*(BSpin(7))$ is computed by

$$Z_{(2)}[c_4, c_6] \otimes (BP^*[c_8]\{1, 2w_4, 2w_8, 2w_4w_8, v_1w_8\} \oplus P(3)^*[c_7]\{c_7\} \oplus P(4)^*[c_7, c_8]\{c_7c_8\}).$$

Therefore we obtain

Corollary 5.9. *Let $w(w_8) = 2$. Then the cohomology $Ih^{*,*}(BSpin(7))$ (resp. $h^{*,*}(BSpin(7); \mathbb{Z}/2)$) is isomorphic to a $\mathbb{Z}/2[\tau]$-submodule of $\mathbb{Z}/2[\tau, c_4, c_6] \otimes A$ (resp. $\mathbb{Z}/2[\tau, c_4, c_6] \otimes B$), where*

$$A = \mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7] \otimes Q(1,2)\{w_4\} \oplus \mathbb{Z}/2[c_7, c_8] \otimes (Q(1,3) - \mathbb{Z}/p)\{b\},$$

$$B = (\mathbb{Z}/2[c_8] \oplus \mathbb{Z}/2[c_7](Q(2) - \mathbb{Z}/p)\{a\} \oplus \mathbb{Z}/2[c_7, c_8](Q(3) - Q(1) + Q_0Q_1 - Q_2)\{c\}$$

with $\quad Q_1b = w_8,\ Q_0a = w_4,\ Q_1Q_0c = w_8,\ Q_2Q_0c = w_4w_8.$

The algebra $BP^*(BSpin(7)) \otimes_{BP^*} Z_{(2)}$ is isomorphic to

$$Z_{(2)}[c_4, c_6, c_8] \otimes (Z_{(2)}\{1, 2w_4, 2w_8, 2w_4w_8\} \oplus \mathbb{Z}/2\{v_1w_8\} \oplus \mathbb{Z}/2[c_7]\{c_7\}).$$

It is known that $2w_2, 2w_8, 2w_4w_8$ are represented by Chern classes but v_1w_8 is not. However, Totaro has shown that the cycle map $\tilde{cl}$ is epic for this case also (see [ScY], [Y3]).

Corollary 5.10. *There is an epimorphism*

$$CH^*(BSpin(7)) \to Z_{(2)}[c_4, c_6, c_8'] \otimes (Z_{(2)}\{1, c_2', c_4', c_6'\} \oplus \mathbb{Z}/2\{\xi_3\} \oplus \mathbb{Z}/2[c_7]\{c_7\}),$$

where c_i' is the i-th Chern class of complexification of the spin representation Δ and ξ_3 is a 6-dimensional element which is not represented by Chern classes. Thus c_2', c_4', c_6' are in Ker(ρ_2) *and $\xi_3 \in$* Ker(ρ).

Next we consider the case $p = odd$. The cases PGL_3 and p_+^{1+2} are easy, and $Ih^{*,*}(BG)$ are given. For example, for $E = p_+^{1+2}$

$$grBP^*(BE) \cong BP^* \otimes H^{even}(BE)/(v_1Q_1H^{odd}(BE)).$$

Finally we consider the case $G = F_4, p = 3$, whose Chow ring is still unknown. The mod 3 cohomology of F_4 is isomorphic to $H^*(BF_4; \mathbb{Z}/3) \cong C \otimes D$ [Tod] with $D = Z_{(3)}[x_{36}, x_{48}]$ and

$$C = \mathbb{Z}/3[x_4, x_8] \otimes \{1, x_{20}, x_{20}^2\} \oplus \mathbb{Z}/3[x_{26}] \otimes \Lambda(x_9) \otimes \{1, x_{20}, x_{21}, x_{25}\},$$

where two terms of C have the intersection $\{1, x_{20}\}$. Then we can prove [KY]

$$grBP^*(BF_4) \cong D \otimes (BP^*\{1, 3x_4\} \oplus BP^* \otimes E \oplus P(3)^*[x_{26}]\{x_{26}\})$$

with $E = Z_{(3)}[x_4, x_8]\{ab | a, b \in \{x_4, x_8, x_{20}\}\}$. Therefore we obtain

Corollary 5.11. *Let $w(E) = 0$ and $w(x_4) = 2$. Then $Ih^{*,*}(BF_4)$ is a $\mathbb{Z}/3[\tau]$-submodule of*

$$D \otimes (\mathbb{Z}/3\{1\} \oplus E \oplus \mathbb{Z}/3[x_{26}] \otimes Q(1,2)\{x_4\})) \otimes \mathbb{Z}/3[\tau].$$

The element $3x_4$ can be proved to be represented by a Chern class, and $x_{26} = Q_2Q_1x_4$. The element x_{36} is also represented by a Chern class, and $P^3x_{36} = x_{48}$. If we can prove that $E/3 \subset \mathrm{Im}(cl_p)$ and $x \in H^{4,3}(BF_4, \mathbb{Z}/3)$, then the above module is just $Ih^{*,*}(BF_4)$ for $p = 3$.

Let G be a simply connected Lie group. Then $H^3(G;\mathbb{Z}) \cong \mathbb{Z}$ and $H^4(G;\mathbb{Z}) \cong 0$. Suppose that $H^*(G;\mathbb{Z})$ has p-torsion. Then it is known that there is an element $x' \in H^3(G;\mathbb{Z})$ such that $0 \neq Q_1x' \in H^{2p+2}(G;\mathbb{Z}/p)$. Taking the classifying space, we get an element $x \in H^4(BG;Z)$ such that $Q_1x \neq 0$ in $H^{2p+3}(BG;\mathbb{Z}/p)$. By Totaro [To2] it is known that $CH^*(BG) \otimes \mathbb{Q} \cong H^*(BG) \otimes \mathbb{Q}$. Hence there is an $s \geq 1$ such that $p^sx_4 \in H^4(BG)$ is in $\mathrm{Im}(cl)$. Thus there is a nonzero element $c \in CH^2(BG)/p$ with $t_{\mathbb{C}}^{2*,*}(c) = 0$. For the groups G_2 or $Spin(7)$ for $p = 2$ and $G = F_4$ for $p = 3$, we can take $s = 1$, since px_4 is represented by the second Chern class c_2.

Proposition 5.12. *Let $p = 2$, 3 or 5. There is a classifying space $B\tilde{G}$ such that for all m, n with $3 \leq n + 1 < m \leq 2n$, the kernel $\mathrm{Ker}(t_{\mathbb{C}}^{m,n})$ is nonzero.*

Proof. Let $\tilde{G} = G \times (\mathbb{Z}/p)^\infty$, where $G = G_2, p = 2$, $G = F_4, p = 3$ or $G = E_8, p = 5$. Recall that $(B\mathbb{Z}/p)^n$ satisfies the Künneth formula for all spaces. For $\mathbb{Z}/p[\tau]$-module generators $x \in H^{*,*}((B\mathbb{Z}/p)^\infty;\mathbb{Z}/p)$, the elements xc are all nonzero and all in $\mathrm{Ker}\, t_{\mathbb{C}}$. □

6. Homotopy category

From the category *Spc*, Voevodsky constructed [Vo1], [Vo2], [MoVo] the ($\mathbb{A}^1$, algebraic) homotopy category *Hot* and the stable homotopy category *SHot*. There are two different types of spheres in *Spc*:

$$S_s^1 = \mathbb{A}^1/\{0,1\} \quad \text{and} \quad S_t^1 = \mathbb{A}^1 - \{0\}. \tag{6.1}$$

The Tate object is $T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1 \cong S_t^1 \wedge S_s^1$ in *Hot*. The category *SHot* is defined by T as the suspension, e.g., $E = \{E_i\}$, $E_i \in Spt$ is a spectrum if there is a map $T \wedge E_i \to E_{i+1}$.

Let Σ_T^∞ be the functor from *Spc* to T-spectra that takes X to $\{T^i \wedge X\}$. If E is a T-spectrum, then the motivic (generalized) cohomology $E^{*,*}(-)$ is defined by

$$E^{m,n}(X) = \mathrm{Hom}_{SHot}(\Sigma_T^\infty(X), S_s^{m-n} \wedge S_t^n \wedge E), \tag{6.2}$$

$$E_{m,n}(X) = \mathrm{Hom}_{SHot}(S_s^{m-n} \wedge S_t^n, \Sigma_T^\infty(X) \wedge E), \tag{6.3}$$

where $\mathrm{Hom}_{SHot}(-,-)$ is the homomorphism defined on *SHot*.

The realization map $t_{\mathbb{C}}$ is originally defined as the functor $t_{\mathbb{C}} : X \to X(\mathbb{C})$ from *Hot* to the category of homotopy spaces. Note that this induces

$$t_{\mathbb{C}} : E^{m,n}(X) \to (t_{\mathbb{C}}E)^m(X(\mathbb{C})). \tag{6.4}$$

The spectrum for the ordinary motivic cohomology is defined as follows. Let $L(X;R)$ for $R = \mathbb{Z}$ or $\mathbb{Z}/p$ be the presheaf sending a connected U to the free R-module generated by the set of all closed irreducible $W \subset U \times X$ such that the projection $W \to U$ is finite and surjective. The Eilenberg-MacLane spectrum is defined as

$$K(R(n), 2n) = L(\mathbb{A}^n;R)/L(\mathbb{A}^n - \{0\};R).$$

Voevodsky proved that $K(R(n), 2n)$ is the Ω-spectrum for the suspension T, namely, $K(R(n), 2n) \cong \Omega_T K(R(n+1), 2n+2)$ in Hot. Define also, for $m < 2n$,

$$K(R(n), m) = \Omega_{S^1}^{2n-m}(R(n), 2n). \tag{6.5}$$

Thus the ordinary motivic cohomology is defined by

$$H^{m,n}(X; R) = \mathrm{Hom}_{Hot}(X, K(R(n), m)). \tag{6.6}$$

Question 6.1. Let $k \subset \mathbb{C}$, and let $0 \neq \tau_n \in H^{n,n}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p)$ (resp. $\tau'_{n+1} \in H^{n+1,n}(K(\mathbb{Z}_{(p)}(n), n+1); \mathbb{Z}/p))$ be the fundamental class (representing the identity map). Then are there isomorphisms

$$h^{2*,*}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}Q_0\tau_n | 0 < i_1 < ... < i_{n-1}],$$
$$h^{2*,*}(K(Z_{(p)}(n), n+1); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}\tau'_{n+1} | 0 < i_1 < ... < i_{n-1}]?$$

It is well known that the dual A_{p*} of the (topological) Steenrod algebra A_p^* is isomorphic to $\mathbb{Z}/p[\xi_1, ...] \otimes \Lambda(\tau_0, ...)$, $|\xi_i| = 2(p^i - 1)$, $|\tau_i| = 2p^i - 1$. Let $P^J \in A_p^*$ (resp. $Q^I \in A_p^*$) be the dual of $\xi_1^{j_1}...$ (resp. $\tau_0^{i_0}...$, $i_k = 0$ or 1), so that $A_p^* \cong \mathbb{Z}/p\{P^J Q^I\}$. Note that $Q^I = \pm Q_0^{i_0}...$. Define $m(J) = \sum_{k=1} j_k$ and $m(I) = \sum_{k=0} i_k$. Then it is also known [Ta] that

$$H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p) \cong \mathbb{Z}/p[Q^I P^J \tau_n | m(I) + 2m(J) < n + i_0]. \tag{6.7}$$

On the other hand, suppose that $Q^I P^J \tau_n \in H^{m,n}(K(\mathbb{Z}/p(n), n); \mathbb{Z}/p)$ for $m \geq 2n$, i.e., $w(Q^I P^J \tau) \leq 0$. Since $w(P^j) = 0$ and $w(Q_i) = -1$, we see that

$$0 \geq w(Q^I P^J \tau_n) = n - m(I).$$

This implies $m(J) = 0, m(I) = n$ and $i_0 \neq 0$. Hence we know that $Q^I P^J \tau$ is the form of the ring generator of the polynomial in the above question.

Remark. Let us write the above as $A = \mathbb{Z}/p[Q_{i_{n-1}}...Q_{i_1}Q_0\tau | 0 < i_1 < ... < i_{n-1}]$. By Tamanoi [Ta], the image $\rho_p(K(\mathbb{Z}/p, n)) = A \subset H^*(K(\mathbb{Z}/p, n); \mathbb{Z}/p)$. Moreover, there is [RWY] the isomorphism $BP^*(K(\mathbb{Z}/p, n)) \otimes_{BP^*} \mathbb{Z}/p \cong A$.

7. Algebraic cobordism

Let BGL denote the infinite Grassmannian, the union of $GL_N(\infty)$ over N. The corresponding generalized cohomology theory is the algebraic K-theory. The motivic cobordism theory $MGL^{*,*}(-)$ is the generalized cohomology theory defined by the Thom spectrum $MGL = \{Th(E_n \to GL_n)\}_n$ identifying $Th(E \oplus O) \cong T \wedge Th(E)$ and $E_n \oplus O \to E_n$ for the trivial line bundle O. It is known (Hu-Kříž [HK], Vezzosi [Ve2]) that

$$MGL^{*,*}((\mathbb{P}^\infty)^n) \cong MGL^{*,*}(pt)[y_1, ..., y_n], \tag{7.1}$$
$$MGL^{*,*}(BGL_n) \cong MGL^{*,*}(pt)[c_1, ..., c_n], \tag{7.2}$$

where the c_i are identified with the elementary symmetric polynomials in the y_i's. Hence the Chern classes are also defined in $MGL^{2*,*}(BG)$. The realization maps

$$t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(BG)_{(p)} \to MU^*(BG)_{(p)}$$

are epic for $G = O(n), SO(4), G_2$ for $p = 2$ and p_+^{1+2} for all primes, because the $MU^*(BG)_{(p)}$ are generated by Chern classes.

For a smooth scheme X over $k \subset \mathbb{C}$, Levine and Morel [LM1], [LM2] constructed the algebraic cobordism theory $\Omega^*(X)$ such that there are natural maps

$$\rho_H : \Omega^*(X) \to H^{2*,*}(X), \qquad \rho_{MGL} : \Omega^*(X) \to MGL^{2*,*}(X) \tag{7.3}$$

with $\rho_H = \rho_{(MGL,H)}\rho_{MGL}$ for the algebraic Thom map $\rho_{(MGL,H)} : MGL^{*,*}(X) \to H^{*,*}(X)$. Moreover, they proved that

$$\rho_H \otimes_{\Omega^*} \mathbb{Z} : \Omega^*(X) \otimes_{\Omega^*} \mathbb{Z} \cong H^{2*,*}(X), \qquad t_{\mathbb{C}}^{2*,*}\rho_{MGL} : \Omega^*(pt) \cong MU^{2*}(pt). \tag{7.4}$$

This implies the motivic version of the Totaro cycle map $\tilde{cl}$:

$$\rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1} : CH^*(X) \to MGL^{2*,*}(X) \otimes_{MGL^{2*,*}} \mathbb{Z}, \tag{7.5}$$

and moreover $t_{\mathbb{C}}^{2*,*}\rho_{MGL}(\rho_H \otimes_{\Omega^*} \mathbb{Z})^{-1}$ is the Totaro cycle map $\tilde{cl}$. Thus the Thom map $\rho_{(MGL,H)}^{2*,*} : MGL^{2*,*}(X) \to H^{2*,*}(X)$ is always epic.

For groups $G = (\mathbb{Z}/p)^n$, $O(n)$, we can easily prove that

$$\Omega^*(BG) \cong MU^*(BG). \tag{7.6}$$

Hence in these cases $MGL^{2*,*}(BG)$ contains $MG^*(BG)$ as a splitting subring.

Corollary 7.1. *Let $\tilde{cl}_p : CH^*(BG)/p \to MU^*(BG) \otimes_{MU^*} \mathbb{Z}/p$ be epic. Then $t_{\mathbb{C}}^{2*,*} : MGL^{2*,*}(X)/p \to MU^*(BG)/p$ is epic, and $\operatorname{Im} \rho_{(MGL,h)} \subset \mathbb{Z}/p[\tau] \otimes h^{2*,*}(X;\mathbb{Z}/p)$, where $\rho_{(MGL,h)} : MGL^{*,*}(X) \to h^{*,*}(X;\mathbb{Z}/p)$ is the induced Thom map.*

The modified cycle maps $\tilde{cl}$ are epic also for the groups $Spin(7)$ for $p = 2$ and PGL_3 for $p = 3$.

By the Thom isomorphism, we get $MGL^{*,*}(BGL) \cong MGL^{*,*}(MGL)$. This means that the Steenrod algebra of $MGL^{*,*}(-)$ is generated as an $MGL^{*,*}(pt)$-module by the Landweber-Novikov operation S_α:

$$MGL^{*,*}(MGL) \cong MGL^{*,*}(pt)\{S_\alpha | \alpha = (i_1, ..., i_n),\ i_j \geq 0\}. \tag{7.7}$$

Here $S_\alpha : MGL^{*,*}(X) \to MGL^{*+2|\alpha|,*+|\alpha|}(X)$ and $|\alpha| = \sum_k i_k k$. These operations satisfy the Cartan formula

$$S_\alpha(xy) = \sum_{\alpha=\beta+\gamma} S_\beta(x)S_\gamma(y), \tag{7.8}$$

and $S_\alpha | MU^*(pt)$ is the usual Landweber-Novikov operation.

Kříž, Hu and Vezzosi construct algebraic Brown-Peterson theory $ABP^{*,*}(-)$ by using a modified Quillen argument. Here we note that we can also construct algebraic BP-theory by using the technique of Novikov(5.4 in [N]). Recall that $MU^*_{(p)} \cong \mathbb{Z}_{(p)}[x_1, ...], |x_i| = -2i$. Define

$$\Delta_{x_i} = \sum_{q\geq 1} (x_i / S_{\Delta_i}(x_i))^{q-1} S_{q\Delta_i}, \tag{7.9}$$

where $\Delta_i = (0, ..., 0, 1, 0, ..., 0)$ (1 in i-th place). Note that $\Delta_{x_i}(x_i) = S_{\Delta_i}(x_i) = 1$ if $i \neq p^j - 1$. Then we can easily prove that $\pi_i = 1 - x_i \Delta_{x_i}$ is a multiplicative projection such that $\pi_i(x_j) = (1 - \delta_{ij})x_j$. Essentially composing (for details, see p. 587 in [N]) the π_i for all $i \neq p^j - 1$, we get the multiplicative projection $\Phi : MGL_{(p)} \to MGL_{(p)}$ such that

$$\Phi(x_i) = \begin{cases} x_i & (\text{if } i = p^j - 1 \text{ for some } j), \\ 0 & (\text{otherwise}). \end{cases} \tag{7.10}$$

Define the algebraic Brown-Peterson spectrum by $\Phi MGL = ABP$. Of course $t_{\mathbb{C}}(ABP) = BP$

Theorem 7.2. *Identify* $BP^* = MU^*_{(p)}/(x_i|i \neq p^j - 1)$. *Then*

$$ABP^{*,*}(X) \cong BP^* \otimes_{MU^*_{(p)}} MGL^{*,*}(X)_{(p)}.$$

Proof. Since $\pi_{x_i}(a) = (1 - x_i\Delta_{x_i})a = a \bmod(x_i)$, we get $\Phi(a) = a \bmod(x_i|i \neq p^j - 1)$ for all $a \in MGL^{*,*}(X)$. The isomorphism is proved, since $ABP^{*,*}(X) \subset MGL^{*,*}(X)_{(p)}$ by the property $\Phi^2 = \Phi$. □

Since $ABP^{*,*}(pt) \cong BP^* \otimes_{MU^*_{(p)}} MGL^{*,*}(pt)$, we can write the above isomorphism as

$$ABP^{*,*}(X) \cong ABP^{*,*} \otimes_{MGL^{*,*}_{(p)}} MGL^{*,*}(X)_{(p)}.$$

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